SEMIREGULARITY AND OBSTRUCTIONS OF COMPLETE INTERSECTIONS

DONATELLA IACONO AND MARCO MANETTI

Abstract. We prove that, on a smooth projective variety over an algebraically closed field of characteristic 0, the semiregularity map annihilates every obstruction to embedded deformations of a local complete intersection subvariety with extendable normal bundle. The proof is based on the theory of L_∞-algebras and Tamarkin-Tsigan calculus on the de Rham complex of DG-schemes.

Introduction

Let X be a smooth algebraic variety, over an algebraically closed field $\mathbb{K}$ of characteristic 0, and let $Z \subset X$ be a locally complete intersection closed subvariety of codimension p. Following [Bl72], the *semiregularity map* $\pi\colon H^1(Z, N_{Z|X}) \to H^{p+1}(X, \Omega_X^{p-1})$, where $N_{Z|X}$ is the normal bundle of Z in X, can be conveniently described by using local cohomology. In fact, since Z is a locally complete intersection, it is well defined the *canonical* cycle class $\{Z\}' \in \Gamma(X, \mathcal{H}^p_Z(\Omega^p_X))$ [Bl72, p. 59] and the contraction with it gives a morphism of sheaves

$$N_{Z|X} \xrightarrow{\lrcorner\{Z\}'} \mathcal{H}^p_Z(\Omega_X^{p-1}).$$

Passing to cohomology, we get a map

$$H^1(Z, N_{Z|X}) \xrightarrow{\lrcorner\{Z\}'} H^1(Z, \mathcal{H}^p_Z(\Omega_X^{p-1})) = H^{p+1}_Z(X, \Omega_X^{p-1}),$$

where the last equality follows from the spectral sequence of local cohomology. Then, the semiregularity map is obtained by taking the composition with the natural map $H^{p+1}_Z(X, \Omega_X^{p-1}) \to H^{p+1}(X, \Omega_X^{p-1})$.

In [Bl72], using Hodge theory and de Rham cohomology, S. Bloch proved that if X is projective, then the semiregularity map annihilates certain obstructions to embedded deformations of Z in X. These obstructions contain in particular the curvilinear ones and, therefore, if the semiregularity map is injective, then the Hilbert scheme of subschemes of X is smooth at Z.

Unfortunately, Bloch's argument is not sufficient to ensure that the semiregularity map annihilates every obstruction to deformations. We have two main reasons to extend Bloch theorem to every obstruction: the first is for testing the power of *derived deformation theory* in a problem where classical deformation theory has failed for almost 40 years. This new approach already worked when Z is a smooth submanifold, see [Ma07, Ia07, Ia11] and Remark 11.2, and the solution of this particular case has given a deep insight about the most appropriate formulation and more useful tools of derived deformation theory. The second reason is related with the theory of reduced Gromov-Witten invariants. Indeed, to define the GW invariants, one needs the virtual fundamental class, defined through an obstruction theory. Whenever the obstruction theory is not carefully chosen, then the virtual fundamental class is zero and the standard GW theory is trivial. A way to overcome this problem, and perform a non trivial Gromov Witten theory is by using a reduced

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obstruction theory, obtained by considering the kernel of a suitable map annihilating obstructions [MPT10, KT11].

The philosophy of derived deformation theory may be summarized in the following way (see e.g. [Ma02]): over a field of characteristic 0, every deformation problem is the classical truncation of an extended deformation problem, which is controlled by a differential graded Lie algebra via Maurer-Cartan equation and gauge equivalence. This differential graded Lie algebra is defined up to quasi-isomorphism and its first cohomology group is equal to the Zariski tangent space of the local moduli space. A morphism of deformation theories is essentially a morphism in the derived category of differential graded Lie algebras (with quasi-isomorphisms as weak equivalences); the induced morphism in cohomology gives, in degrees 1 and 2, the tangent and obstruction map, respectively.

Clearly, a morphism from a deformation theory into an unobstructed deformation theory provides an obstruction map annihilating every obstruction. Using this basic principle, we are able to prove that the semiregularity map annihilates every obstruction under the following additional assumption:

Set-up: *Z is closed of codimension p in X and there exists a Zariski open subset $U \subset X$ and a vector bundle $E \to U$ of rank p such that $Z \subset U$ and Z is the zero locus of a section $f \in \Gamma(U, E)$.*

This is obviously satisfied for complete intersections of hypersurfaces, while for Z of codimension 2 we refer to [OSS80] for a discussion about the validity of the above set-up.

Our main results, proved entirely with deformation theory techniques, are summarized in the next theorem, where $\mathbb{H}^*$ denotes hypercohomology groups.

Main Theorem. *Let X be a smooth algebraic variety, over an algebraically closed field of characteristic 0, and $Z \subset X$ a closed subvariety of codimension p as in the above set-up. Then, the composition of the semiregularity map and the truncation*

$$H^1(Z, N_{Z|X}) \xrightarrow{\pi} H^{p+1}(X, \Omega_X^{p-1}) \xrightarrow{\tau} \mathbb{H}^{2p}(X, \Omega_X^0 \xrightarrow{d} \cdots \xrightarrow{d} \Omega_X^{p-1})$$

annihilates every obstruction to infinitesimal embedded deformations of Z in X.

Clearly, if the truncation map $H^{p+1}(X, \Omega_X^{p-1}) \xrightarrow{\tau} \mathbb{H}^{2p}(X, \Omega_X^0 \xrightarrow{d} \cdots \xrightarrow{d} \Omega_X^{p-1})$ is injective then the semiregularity map annihilates every obstruction too. A sufficient condition for the injectivity of τ is the degeneration at level E_1 of the Hodge-de Rham spectral sequence; in particular, this is true whenever X is smooth proper over a field of characteristic 0 [GH78, DI87].

Corollary. *In the same assumption of the main theorem, if the Hodge-de Rham spectral sequence of X degenerates at level E_1, then the semiregularity map*

$$H^1(Z, N_{Z|X}) \xrightarrow{\pi} H^{p+1}(X, \Omega_X^{p-1})$$

annihilates every obstruction to infinitesimal embedded deformations of Z in X.

The set-up assumption is purely technical and there is no reason for the failure of the theorem for general local complete intersection subvarieties.

The two underlying ideas used in the proof of the above theorem are:

(1) to give a purely algebraic proof for smooth submanifolds using the ideas of [IM10] and to extend it to the case of DG-schemes, considered in the sense of I. Ciocan-Fontanine and M. Kapranov [Ka01, CK01];
(2) to replace the embedding $Z \subset X$ with a quasi-isomorphic embedding of smooth DG-schemes.

For the second idea it is necessary to assume the existence of the bundle E and without this assumption the approach of DG-schemes seems insufficient. A possible and natural way to overcome this difficulty is to investigate more deeply the problem in the framework

of derived algebraic geometry. After the appearence of the first version of this paper an interesting contribution in this direction has been performed by Pridham [Pr12].

The main theorem, as well as its proof, suggests that, from the point of view of deformation theory, the semiregularity map π is not the correct object to study and should be replaced by its composition with τ; this partially explains the past difficulties to relate deformations and semiregularity and to describe it as a component of the differential of a morphism of deformation theories. The vector spaces $\mathbb{H}^{2p-1}(X,\Omega^0_X\xrightarrow{d}\cdots\xrightarrow{d}\Omega^{p-1}_X)$ and $\mathbb{H}^{2p}(X,\Omega^0_X\xrightarrow{d}\cdots\xrightarrow{d}\Omega^{p-1}_X)$ may be interpreted as the tangent and obstruction spaces of the p-th intermediate Jacobian of X, respectively, and $\tau\pi$ as a component of the differential of the (extended) infinitesimal Abel-Jacobi map [BF03, Pr12, FiMa].

The assumption for the base field $\mathbb{K}$ to be of characteristic 0 is essential, both for the method of the proofs and the validity of the main results. For simplicity and for giving precise references (especially to [Ar76, Bl72, Se06]) we also assume that $\mathbb{K}$ is algebraically closed, although this assumption is not strictly necessary.

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0.1. **Content of the paper.** The first 3 sections contain a list of algebraic preliminaries: we recall the notion and the main properties of semifree modules, homotopy fibers of morphisms of differential graded Lie algebras and the semicosimplicial Thom-Whitney-Sullivan construction. In Section 4, we reinterpret some results of Hinich, Fiorenza, Iacono and Martinengo [Hin97, FIM09] in terms of descent theory of reduced Deligne groupoids. In Section 5, we describe two differential graded Lie algebras controlling the embedded deformations of Z as in the above set-up: they can be seen as spaces of derived sections of some sheaves of derivation of a DG-scheme. In Sections 6 and 7, we shall prove that L_∞ morphisms, Tamarkin-Tsigan calculus and Cartan homotopies commutes with Thom-Whitney-Sullivan construction, as well as the L_∞ morphism associated with a Cartan homotopy. In Section 8, we prove that Tamarkin-Tsigan calculus holds also for the algebraic de Rham complex of a DG-scheme, while, in Section 9, we prove some base change results about local cohomology of quasi-coherent DG-sheaves over DG-schemes. Finally, Sections 10 and 11 are devoted to the proof of the main theorem, using all the result of the previous sections.

0.2. **Notation.** Throughout the paper, we work over an algebraically closed field $\mathbb{K}$ of characteristic zero. All vector spaces, linear maps, tensor products etc. are intended over $\mathbb{K}$, while every graded object is considered graded over $\mathbb{Z}$.

A DG-vector space V is the data of a $\mathbb{Z}$-graded vector space, $V=\bigoplus_{n\in\mathbb{Z}}V^n$ together with a differential $\delta\colon V\to V$ of degree $+1$. For every homogeneous element $v\in V^i$, we denote by $\overline{v}=i$ its degree. As usual, we use the standard notation for cocycles, coboundaries and cohomology groups: $Z(V)=\ker\delta, B(V)=\operatorname{Im}\delta$ and $H(V)=Z(V)/B(V)$, respectively. For any integer n and any DG-vector space (V,δ), we define the shifted DG-vector space $(V[n],\delta_{V[n]})$, where $V[n]^i=V^{i+n}$ and $\delta_{V[n]}=(-1)^n\delta$.

Given two DG vector spaces V and W, we denote by $\operatorname{Hom}^n_{\mathbb{K}}(V,W)$ the space of $\mathbb{K}$-linear map $f:V\to W$, such that $f(V^i)\subset W^{i+n}$, for every $i\in\mathbb{Z}$. Then $\operatorname{Hom}^*_{\mathbb{K}}(V,W)=\oplus_{n\in\mathbb{Z}}\operatorname{Hom}^n_{\mathbb{K}}(V,W)$ has a natural structure of DG-vector space with differential $\delta(f)=\delta_W\circ f-(-1)^{\overline{f}}f\circ\delta_V$.

For a given graded vector space V, we will use both symbols $\operatorname{Sym}^n(V)$ and $\bigodot^n V$ for denoting its graded symmetric n-th power. The direct sum of all symmetric powers carries a natural structure of graded algebra and also a natural structure of graded coalgebra. For the clarity of exposition, we will adopt the following convention:

(1) $\mathrm{Sym}^*(V)=\bigoplus_{n\geq 0}\mathrm{Sym}^n(V)$ is the graded symmetric algebra generated by V;
(2) $\bigodot^* V=\bigoplus_{n\geq 0}\bigodot^n V$ is the graded symmetric coalgebra generated by V;
(3) $\overline{\bigodot}^* V=\bigoplus_{n\geq 1}\bigodot^n V$ is the reduced graded symmetric coalgebra generated by V.

In this paper, we use several kind of cohomology groups/sheaves. Unless otherwise specified, we shall denote by:

- $H^n(X,\mathcal{F})$ the n-th cohomology group of a sheaf $\mathcal{F}$ on X;
- $\mathbb{H}^n(X,\mathcal{F}^*)$ the n-th hypercohomology group of a complex of sheaves $\mathcal{F}^*$;
- $\mathcal{H}^n_Z(X,\mathcal{F})$ the n-th cohomology sheaf of a sheaf $\mathcal{F}$, with support in $Z\subset X$ [Gr67];
- $\mathbb{H}^n_Z(X,\mathcal{F}^*)$ the n-th hypercohomology group with support in $Z\subset X$ of a complex of sheaves $\mathcal{F}^*$;
- $\mathcal{H}^n(\mathcal{F}^*)$ the n-th cohomology sheaf of a complex of sheaves $\mathcal{F}^*$;

1. Review of semifree modules over DG-rings

By a DG-ring we mean a unitary graded commutative ring $A=\bigoplus_{n\in\mathbb{Z}}A^n$ endowed with a differential $\delta\colon A\to A$ of degree $+1$. Graded commutativity means that $ab=(-1)^{\overline{a}\,\overline{b}}ba$ and the graded Leibniz rule is $\delta(ab)=(\delta a)b+(-1)^{\overline{a}}a(\delta b)$. Notice that

$$\cdots\xrightarrow{\delta}A^{i-2}\xrightarrow{\delta}A^{i-1}\xrightarrow{\delta}A^i\xrightarrow{\delta}\cdots$$

is a complex of $Z^0(A)$-modules; in particular, every cohomology group $H^i(A)$ is a $Z^0(A)$-module. A DG-ring which is also a DG-vector space will be called a DG-algebra.

Given a DG-ring A, a DG-module over A is a $\mathbb{Z}$-graded A-module $M=\bigoplus_{n\in\mathbb{Z}}M^n$, endowed with a differential $\delta\colon M^i\to M^{i+1}$, satisfying the conditions:

(1) $am=(-1)^{\overline{a}\,\overline{m}}ma$,
(2) $\delta(am)=(\delta a)m+(-1)^{\overline{a}}a(\delta m)$,

for every pair of homogeneous elements $a\in A$ and $m\in M$. As above, the sequence

$$\cdots\xrightarrow{\delta}M^{i-1}\xrightarrow{\delta}M^i\xrightarrow{\delta}M^{i+1}\xrightarrow{\delta}\cdots$$

is a complex of $Z^0(A)$-modules; then, also the cohomology groups $H^i(M)$ are $Z^0(A)$-modules.

A morphism of DG-modules is an A-linear map of degree 0 commuting with differentials; a quasi-isomorphism is a morphism inducing an isomorphism in cohomology.

Given two DG-modules M and N over A, their tensor product $M\otimes_A N$ is defined as the quotient of $M\otimes_{\mathbb{Z}}N$ by the graded submodule generated by all the elements $ma\otimes n-m\otimes an$, for every $m\in M, n\in N$ and $a\in A$; notice that, the degree of $n\otimes m$ is $\overline{n}+\overline{m}$. Let (M,δ_M) and (N,δ_N) be two DG-modules. We define the graded Hom as the graded vector space $\mathrm{Hom}^*_A(M,N)=\oplus_{n\in\mathbb{Z}}\mathrm{Hom}^n_A(M,N)$, where

$$\mathrm{Hom}^n_A(M,N)=\{\phi\colon M\to N\ \mid \phi(M^i)\subset N^{i+n},\ \phi(ma)=\phi(m)a,\ i\in\mathbb{Z},\ m\in M,\ a\in A\}.$$

Note that $\mathrm{Hom}^*_A(M,N)$ is a DG-module, with left multiplication $(a\phi)(m)=a\phi(m)$, for all $a\in A$ and $m\in M$, and differential $\delta'\colon\mathrm{Hom}^n_A(M,N)\to\mathrm{Hom}^{n+1}_A(M,N)$, given by $\delta'(\phi)=\delta_N\circ\phi-(-1)^n\phi\circ\delta_M$ [Ma04b, Chapter IV.3].

In order to simplify the terminology, from now on the term A-modules will mean a DG-module over a DG-ring A.

A homotopy between two morphisms $f,g\colon M\to N$ of A-modules is a morphism $h\in\mathrm{Hom}^{-1}_A(M,N)$, such that $\delta_N\circ h+h\circ\delta_M=f-g$. Homotopic morphisms induce the same morphism in cohomology.

Definition 1.1. Let A be a DG-ring, an A-module F is called *semifree* if $F = \oplus_{i\in I} Am_i$, $\overline{m_i} \in \mathbb{Z}$ and there exists a filtration $\emptyset = I(0) \subset I(1) \subset \cdots \subset I(n) \subset \cdots$ such that

$$I = \cup_n I(n), \qquad i \in I(n+1) \Rightarrow \delta m_i \in \oplus_{i\in I(n)} Am_i.$$

Example 1.2. If $F = \oplus_{i\in I} Am_i$ and the set of degrees $\{\overline{m_i}\}$ is bounded from above, then F is semifree: in fact if $t = \max\{\overline{m_i}\}$ it is sufficient to choose $I(h) = \{i \mid \overline{m_i} > t - h\}$.

Example 1.3. Let $B = A[x_1, \ldots, x_n]$, with every x_i of degree ≤ 0, then B is a semifree as A-module, for every differential. In fact $B \simeq \oplus_{m\in I} Am$, where I is the set of monomials in the variables x_i.

The next results are quite standard and easy to prove (see e.g. [AF91]):

(1) Every quasi-isomorphism of semifree A-modules is a homotopy equivalence.
(2) If F is semifree and $M \to N$ is a quasi-isomorphism of A-modules, then also the induced morphism $M \otimes_A F \to N \otimes_A F$ is a quasi-isomorphism.
(3) Every A-module admits a semifree resolution, i.e., for every A-module M there exists a semifree module F and a surjective quasi-isomorphism $F \to M$.

Given a morphism of DG-ring $A \to B$ and a B-module M, we define $\operatorname{Der}^*_A(B, M) = \bigoplus_{n\in\mathbb{Z}} \operatorname{Der}^n_A(B, M)$, where

$$\operatorname{Der}^n_A(B, M) = \{\phi \in \operatorname{Hom}^n_A(B, M) \mid \phi(ab) = \phi(a)b + (-1)^{n\overline{a}} a\phi(b)\}.$$

As for $\operatorname{Hom}^*_B(M, N)$, there exists a structure of B-module on $\operatorname{Der}^*_A(B, M)$, with the analogous left multiplication $(a\phi)(b) = a\phi(b)$, for all $a, b \in B$, and analogous differential $\delta' : \operatorname{Der}^n_A(B, M) \to \operatorname{Der}^{n+1}_A(B, M)$, given by $\delta'(\phi) = \delta_M \circ \phi - (-1)^n \phi \circ \delta_B$.

Given a morphism of DG-rings $A \to B$, we denote by $\Omega_{B/A}$ the module of relative Kähler differentials of B over A and by $d\colon B \to \Omega_{B/A}$ the universal derivation, determined by the property that for every B-module M the composition with d gives an isomorphism of B-modules

$$\boldsymbol{i}\colon \operatorname{Der}^*_A(B, M) \xrightarrow{\simeq} \operatorname{Hom}^*_B(\Omega_{B/A}, M), \qquad \boldsymbol{i}_\alpha(db) = \alpha(b) \quad \alpha \in \operatorname{Der}^*_A(B, M), b \in B.$$

The construction of $\Omega_{B/A}$ is essentially the same as in the non graded case, the differential on $\Omega_{B/A}$ commutes with the differential of B and $d \in \operatorname{Der}^0_A(B, \Omega_{B/A})$.

For later use, we point out that in the situation of Example 1.3 we have $\Omega_{B/A} = \oplus B dx_i$, $\delta(dx_i) = d(\delta x_i)$ and then $\Omega_{B/A}$ is a semifree B-module.

2. Homotopy fiber of a morphism of differential graded Lie algebras

A *differential graded Lie algebra* is the data of a differential graded vector space (L, d) together with a bilinear map $[-,-]\colon L \times L \to L$ (called bracket) of degree 0 such that the following conditions are satisfied:

(1) (graded skewsymmetry) $[a, b] = -(-1)^{\overline{a}\,\overline{b}}[b, a]$.
(2) (graded Jacobi identity) $[a, [b, c]] = [[a, b], c] + (-1)^{\overline{a}\,\overline{b}}[b, [a, c]]$.
(3) (graded Leibniz rule) $d[a, b] = [da, b] + (-1)^{\overline{a}}[a, db]$.

In particular, the Leibniz rule implies that the bracket of a DGLA L induces a structure of graded Lie algebra on its cohomology $H^*(L) = \oplus_i H^i(L)$. Moreover, a DGLA is *abelian* if its bracket is trivial.

A *morphism* of differential graded Lie algebras $\chi\colon L \to M$ is a linear map that preserves degrees and commutes with brackets and differentials.

A *quasi-isomorphism* of DGLAs is a morphism that induces an isomorphism in cohomology. Two DGLAs L and M are said to be *quasi-isomorphic*, or *homotopy equivalent*, if they are equivalent under the equivalence relation generated by: $L \sim M$ if there exists

a quasi-isomorphism $\chi\colon L\to M$. A differential graded Lie algebra is *homotopy abelian* if it is quasi-isomorphic to an abelian DGLA.

The *homotopy fiber* of a morphism of DGLAs $\chi\colon L\to M$ is the differential graded Lie algebra

$$TW(\chi):=\{(l,m(t,dt))\in L\times M[t,dt]\ \mid\ m(0,0)=0,\, m(1,0)=\chi(l).\}$$

Lemma 2.1. *Let $\chi\colon L\to M$ be a morphism of differential graded Lie algebras such that:*

(1) *$\chi\colon L\to M$ is injective,*
(2) *$\chi\colon H^*(L)\to H^*(M)$ is injective.*

Then, the homotopy fiber is homotopy abelian.

Proof. This result has been proved in [IM10] using L_∞-algebras; here we sketch a more elementary proof. The same argument used in [IM10, Proposition 3.4] shows that that there exists a direct sum decomposition $M=\chi(L)\oplus V$ as a direct sum of differential graded vector space and, therefore, the mapping cone $L\oplus M[-1]$ of χ is quasi-isomorphic to $V[-1]$. Consider $V[-1]$ as an abelian differential graded Lie algebras and consider the morphism of DGLAs

$$f\colon V[-1]\to TW(\chi),\qquad f(v)=(0,dt\,v).$$

Since the map

$$TW(\chi)\to L\oplus M[-1],\qquad (l,p(t)m_1+q(t)dtm_2)\mapsto (l,\int_0^1 q(t)dtm_2),$$

is a quasi-isomorphism of complexes, it follows that f is a quasi-isomorphism of DGLAs. □

Remark 2.2. Assume that $\chi\colon L\to M$ is an injective morphism of DGLAs, then its cokernel $M/\chi(L)$ is a differential graded vector space and the map

$$TW(\chi)\to (M/\chi(L))[-1],\qquad (l,p(t)m_0+q(t)dtm_1)\mapsto \left(\int_0^1 q(t)dt\right)m_1\pmod{\chi(L)},$$

is a surjective quasi-isomorphism.

Example 2.3. Let W be a differential graded vector space and $\gamma\in W$ a cocycle with non trivial cohomology class. Then, the inclusion

$$\chi\colon\{f\in \operatorname{Hom}^*_{\mathbb{K}}(W,W)\mid f(\gamma)=0\}\to \operatorname{Hom}^*_{\mathbb{K}}(W,W)$$

satisfies the hypothesis of Lemma 2.1. Therefore, the DGLA $TW(\chi)$ is homotopy abelian. In fact, the morphism of complexes $\mathbb{K}\gamma\to W$ is injective in cohomology and then by Künneth formula the map $\operatorname{Hom}^*_{\mathbb{K}}(W,W)\to \operatorname{Hom}^*_{\mathbb{K}}(\mathbb{K}\gamma,W)$ is surjective in cohomology.

3. Semicosimplicial Thom-Whitney-Sullivan construction

Let $\mathbf{\Delta}_{\mathrm{mon}}$ be the category whose objects are the finite ordinal sets $[n]=\{0,1,\ldots,n\}$, $n=0,1,\ldots$, and whose morphisms are order-preserving injective maps among them. Every morphism in $\mathbf{\Delta}_{\mathrm{mon}}$, different from the identity, is a finite composition of *coface* morphisms:

$$\partial_k\colon[i-1]\to[i],\qquad \partial_k(p)=\begin{cases}p & \text{if } p<k\\ p+1 & \text{if } k\le p\end{cases},\qquad k=0,\ldots,i.$$

The relations about compositions of them are generated by

$$\partial_l\partial_k=\partial_{k+1}\partial_l\,,\qquad \text{for every } l\le k.$$

According to [EZ50, We94], a *semicosimplicial* object in a category $\mathbf{C}$ is a covariant functor $A^\Delta\colon \mathbf{\Delta}_{\text{mon}} \to \mathbf{C}$. Equivalently, a semicosimplicial object A^Δ is a diagram in $\mathbf{C}$:

$$A_0 \rightrightarrows A_1 \rightrightarrows\!\!\!\!\!\!\rightarrow A_2 \rightrightarrows\!\!\!\!\!\!\rightrightarrows \cdots,$$

where each A_i is in $\mathbf{C}$, and, for each $i > 0$, there are $i+1$ morphisms

$$\partial_k\colon A_{i-1} \to A_i, \qquad k = 0, \ldots, i,$$

such that $\partial_l \partial_k = \partial_{k+1}\partial_l$, for any $l \leq k$.

Given a semicosimplicial differential graded vector space

$$V^\Delta: \qquad V_0 \rightrightarrows V_1 \rightrightarrows\!\!\!\!\!\!\rightarrow V_2 \rightrightarrows\!\!\!\!\!\!\rightrightarrows \cdots,$$

the graded vector space $\prod_{n\geq 0} V_n[-n]$ has two differentials

$$d = \sum_n (-1)^n d_n, \qquad \text{where} \quad d_n \text{ is the differential of } V_n,$$

and

$$\partial = \sum_i (-1)^i \partial_i, \qquad \text{where} \quad \partial_i \text{ are the coface maps.}$$

More explicitly, if $v \in V_n^i$, then the degree of v is $i+n$ and

$$d(v) = (-1)^n d_n(v) \in V_n^{i+1}, \qquad \partial(v) = \partial_0(v) - \partial_1(v) + \cdots + (-1)^{n+1}\partial_{n+1}(v) \in V_{n+1}^i.$$

Since $d\partial + \partial d = 0$, we define $\operatorname{tot}(V^\Delta)$ as the graded vector space $\prod_{n\geq 0} V_n[-n]$, endowed with the differential $d + \partial$.

Let V^Δ be a semicosimplicial differential graded vector space and $(A_{PL})_n$ the differential graded commutative algebra of polynomial differential forms on the standard n-simplex $\{(t_0,\ldots,t_n) \in \mathbb{K}^{n+1} \mid \sum t_i = 1\}$ [FHT01]:

$$(A_{PL})_n = \frac{\mathbb{K}[t_0,\ldots,t_n,dt_0,\ldots,dt_n]}{(1-\sum t_i, \sum dt_i)}.$$

For every n, m the tensor product $(A_{PL})_m \otimes V_n$ is a differential graded vector space and then also $\prod_n (A_{PL})_n \otimes V_n$ is a differential graded vector space.

Denoting by

$$\delta^k\colon (A_{PL})_n \to (A_{PL})_{n-1}, \quad \delta^k(t_i) = \begin{cases} t_i & \text{if } 0 \leq i < k \\ 0 & \text{if } i = k \\ t_{i-1} & \text{if } k < i \end{cases}, \qquad k = 0, \ldots, n,$$

the face maps, for every $0 \leq k \leq n$, there are well-defined morphisms of differential graded vector spaces

$$\delta^k \otimes Id\colon (A_{PL})_n \otimes V_n \to (A_{PL})_{n-1} \otimes V_n,$$
$$Id \otimes \partial_k\colon (A_{PL})_{n-1} \otimes V_{n-1} \to (A_{PL})_{n-1} \otimes V_n.$$

The Thom-Whitney-Sullivan differential graded vector space of V^Δ is denoted by $TW(V^\Delta) \subset \prod_n (A_{PL})_n \otimes V_n$ and is the graded subspace whose elements are the sequences $(x_n)_{n\in\mathbb{N}}$ satisfying the equations

$$(\delta^k \otimes Id)x_n = (Id \otimes \partial_k)x_{n-1}, \quad \text{for every } \ 0 \leq k \leq n.$$

In [Whi57], Whitney noted that the integration maps

$$\int_{\Delta^n} \otimes \operatorname{Id}\colon (A_{PL})_n \otimes V_n \to \mathbb{K}[-n] \otimes V_n = V_n[-n]$$

give a quasi-isomorphism of differential graded vector spaces

$$I\colon (TW(V^\Delta), d_{TW}) \to (\operatorname{tot}(V^\Delta), d_{\text{tot}}).$$

Further details can be found in [NaA87, Get04, FiMa07, CG08].

Example 3.1. Let $\mathcal{L}$ be a sheaf of differential graded vector spaces over an algebraic variety X and $\mathcal{U} = \{U_i\}$ an open cover of X; assume that the set of indices i is totally ordered. Then, we can define the semicosimplicial DG vector space of Čech alternating cochains of $\mathcal{L}$ with respect to the cover $\mathcal{U}$:

$$\mathcal{L}(\mathcal{U}): \qquad \prod_i \mathcal{L}(U_i) \rightrightarrows \prod_{i<j} \mathcal{L}(U_{ij}) \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \prod_{i<j<k} \mathcal{L}(U_{ijk}) \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} \cdots .$$

Clearly, in this case, the total complex $\operatorname{tot}(\mathcal{L}(\mathcal{U}))$ is the associated Čech complex $C^*(\mathcal{U}, \mathcal{L})$. We will denote by $TW(\mathcal{U}, \mathcal{L})$ the associated Thom-Whitney complex. The integration map $TW(\mathcal{U}, \mathcal{L}) \to C^*(\mathcal{U}, \mathcal{L})$ is a surjective quasi-isomorphism. If $\mathcal{L}$ is a quasi-coherent DG-sheaf and every U_i is affine, then the cohomology of $TW(\mathcal{U}, \mathcal{L})$ is the same of the cohomology of $\mathcal{L}$.

Example 3.2. Let

$$\mathfrak{g}^{\Delta}: \qquad \mathfrak{g}_0 \rightrightarrows \mathfrak{g}_1 \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \mathfrak{g}_2 \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} \cdots ,$$

be a semicosimplicial differential graded Lie algebra, i.e., each $\mathfrak{g}_i$ is a DGLA each ∂_k is a morphism of DGLAs. Then, in this case too, we can apply the Thom-Whitney construction: it is evident $TW(\mathfrak{g}^{\Delta})$ has a structure of a differential graded lie algebra.

Example 3.3. Let $\chi : L \to M$ be a morphism of differential graded Lie algebras. Then, we can consider the semicosimplicial DGLA

$$\chi^{\Delta}: \qquad L \rightrightarrows M \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} 0 \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} \cdots , \qquad \text{with} \quad \partial_0 = \chi \text{ and } \partial_1 = 0.$$

It turns out [Ma07, FiMa07] that the total complex $\operatorname{tot}(\chi^{\Delta})$ coincides with the mapping cone of χ, i.e.,

$$\operatorname{tot}(\chi^{\Delta})^i = L^i \oplus M^{i-1}, \qquad d(l, m) = (dl, \chi(l) - dm),$$

and the Thom-Whitney-Sullivan construction coincides with the homotopy fiber of χ:

$$TW(\chi^{\Delta}) = TW(\chi) = \{(l, m(t, dt)) \in L \times M[t, dt] \mid m(0, 0) = 0,\, m(1, 0) = \chi(l)\}.$$

Whenever $\mathfrak{g}^{\Delta}$ is a semicosimplicial differential graded Lie algebra, we have just noticed that $TW(\mathfrak{g}^{\Delta})$ is a differential graded Lie algebra. Moreover, every $\mathfrak{g}_i$ is a DGLA and so, in particular, a differential graded vector space, thus we can consider the total complex $\operatorname{tot}(\mathfrak{g}^{\Delta})$. It turns out that the complex $\operatorname{tot}(\mathfrak{g}^{\Delta})$ has no natural DGLA structure, even in the easy case of Example 3.3 of a morphism of DGLAs [IM10, Example 2.7].

Lemma 3.4. *Let $\mathfrak{g}^{\Delta}$ be a semicosimplicial DGLA, L a DGLA and $\varphi\colon L \to \mathfrak{g}_0$ a morphism of DGLAs, such that $\partial_0 \circ \varphi = \partial_1 \circ \varphi$. Then it is well defined a morphism of DGLAs $h\colon L \to TW(\mathfrak{g}^{\Delta})$ giving a commutative diagram*

$$\begin{tikzcd} L \arrow[r, "h"] \arrow[dr, "\psi"'] & TW(\mathfrak{g}^{\Delta}) \arrow[d, "I"] \\ & \operatorname{tot}(\mathfrak{g}^{\Delta}), \end{tikzcd}$$

where $\psi\colon L \to \operatorname{tot}(\mathfrak{g}^{\Delta})$ is the composition of φ with the inclusion $\mathfrak{g}_0 \subset \operatorname{tot}(\mathfrak{g}^{\Delta})$.

Proof. A straightforward computation shows that the map $h\colon L \to TW(\mathfrak{g}^{\Delta})$ defined as

$$h(l) = (1 \otimes \varphi(l), 1 \otimes \partial_0(\varphi(l)), 1 \otimes \partial_0^2(\varphi(l)), \ldots, 1 \otimes \partial_0^n(\varphi(l)), \ldots),$$

is a well defined morphism of differential graded Lie algebras.

Since I contracts the polynomial differential forms in $(A_{PL})_n$ by integrating over the standard simplex Δ_n, we have that, $I(h(l)) = \varphi(l) \in \mathfrak{g}_0^i$, for every $l \in L^i$. □

4. Descent of formal reduced Deligne groupoid

Denote by **Set** the category of sets (in a fixed universe) and by **Grpd** the category of small groupoids. We shall consider **Set** as a full subcategory of **Grpd**. Given a small groupoid G it is convenient, for our purposes, to think to their objects and arrows as vertices and edges of its nerve, respectively; therefore, G_0 will denote the set of objects and $G_1(x,y)$ the set of morphisms from x to y.

Finally, denote by **Art** the category of local Artin $\mathbb{K}$-algebras with residue field $\mathbb{K}$. Unless otherwise specified, for every object $A \in \mathbf{Art}$, we denote by $\mathfrak{m}_A$ its maximal ideal.

Let $\mathbf{C}$ be a category with a final object $*$, throughout this paper, by a formal object in $\mathbf{C}$, we shall mean a covariant functor $F\colon \mathbf{Art} \to \mathbf{C}$ such that $F(\mathbb{K}) = *$. In particular, a formal groupoid is a functor $G\colon \mathbf{Art} \to \mathbf{Grpd}$ such that $G(\mathbb{K}) = *$, and a formal set is a functor $G\colon \mathbf{Art} \to \mathbf{Set}$ such that $G(\mathbb{K}) = *$ (these last ones are also called functors of Artin rings).

Definition 4.1. Let L be a fixed nilpotent differential graded Lie algebra. The Maurer-Cartan set associated with L is

$$\mathrm{MC}(L) = \left\{ x \in L^1 \;\middle|\; dx + \frac{1}{2}[x,x] = 0 \right\}.$$

The gauge action $* : \exp(L^0) \times \mathrm{MC}_L \longrightarrow \mathrm{MC}_L$ is defined by the explicit formula

$$e^a * x := x + \sum_{n \geq 0} \frac{[a,-]^n}{(n+1)!}([a,x] - da).$$

Finally, the *Deligne groupoid* associated with L is the groupoid $\mathrm{Del}(L)$ defined as follows:

(1) the objects are $\mathrm{Del}(L)_0 = \mathrm{MC}(L)$,
(2) the morphisms are $\mathrm{Del}(L)_1(x,y) = \{e^a \in \exp(L^0) \mid e^a * x = y\}$, for $x, y \in \mathrm{Del}(L)_0$.

For each $x \in \mathrm{MC}(L)$ the *irrelevant stabilizer* of x is defined as the subgroup:

$$I(x) = \{e^{dh+[x,h]} | \, h \in L^{-1}\} \subset \mathrm{Del}(L)_1(x,x).$$

Note that, since $e^{dh+[x,h]} * x = x$, the irrelevant stabilizer $I(x)$ is contained in the stabilizer of x under the gauge action. Moreover, $I(x)$ is a normal subgroup of the stabilizer of x, since for any $a \in L^0$ we have (see e.g. [Kon94, Ma07, Ye11])

$$e^a I(x) e^{-a} = I(y), \qquad \text{with} \qquad y = e^a * x. \tag{4.1}$$

The above formula implies also that, for every $x, y \in MC(L)$, we have a natural isomorphism

$$\frac{\mathrm{Del}(L)_1(x,y)}{I(x)} = \frac{\mathrm{Del}(L)_1(x,y)}{I(y)},$$

with $I(x)$ and $I(y)$ acting in the obvious way.

Definition 4.2 ([Kon94, Ye11])**.** The *reduced Deligne groupoid* associated with a nilpotent differential graded Lie algebra L is the groupoid $\overline{\mathrm{Del}}(L)$ having the same objects as $\mathrm{Del}(L)$ and as morphisms

$$\overline{\mathrm{Del}}(L)_1(x,y) := \frac{\mathrm{Del}(L)_1(x,y)}{I(x)} = \frac{\mathrm{Del}(L)_1(x,y)}{I(y)}.$$

Example 4.3. Let B, C be two $\mathbb{K}$-algebras, $\{x_i\}$ a set of indeterminates of non positive degree and $R = B[\{x_i\}] \to C$ a quasi-isomorphism of DG-algebras and denote by $\delta \in \mathrm{Der}^1_B(R,R)$ the differential of R. Notice that as a graded algebra, R is a free B-algebra with generators x_i of degree ≤ 0. Moreover, $R^i = 0$ for every $i > 0$, $H^0(R,\delta) = C$ and

$H^i(R,\delta)=0$, for every $i\neq 0$. Given a local Artin $\mathbb{K}$-algebra A, let us describe the Deligne groupoid and the irrelevant stabilizers of the nilpotent DGLA $L=\operatorname{Der}^*_B(R,R)\otimes\mathfrak{m}_A$.

There is a natural bijection between Maurer-Cartan elements and differentials $\rho'\colon R\otimes A\to R\otimes A$, which are $B\otimes A$ linear and equal to δ modulus $\mathfrak{m}_A$, while the set of morphisms, in the Deligne groupoid, between ρ and ρ' is the set of $B\otimes A$-linear isomorphisms $(R\otimes A,\rho)\to(R\otimes A,\rho')$, which are the identity modulus $\mathfrak{m}_A$.

Lemma 4.4. *In the notation above, the irrelevant stabilizer $I(\rho)$ is the group of $B\otimes A$-linear automorphisms of the DG-algebra $(R\otimes A,\rho)$, which are the identity modulus $\mathfrak{m}_A$ and inducing the identity in cohomology.*

Proof. Let $a\in\operatorname{Der}^0_B(R,R)\otimes\mathfrak{m}_A$, then by definition $e^a\in I(\rho)$ if and only if there exists $b\in\operatorname{Der}^{-1}_B(R,R)\otimes\mathfrak{m}_A$ such that $a=[\rho,b]$. On the other hand, e^a is an isomorphism of $(R\otimes A,\rho)$ if and only if $[\rho,a]=0$, and induces the identity in cohomology if and only if a induces the trivial map in cohomology. Therefore, we only need to prove that if $a\colon R\otimes A\to R\otimes A$ is a morphism of complexes which is trivial in cohomology then $a=\rho b+b\rho$ for some $b\in\operatorname{Der}^{-1}_B(R,R)\otimes\mathfrak{m}_A=\operatorname{Der}^{-1}_B(R,R\otimes\mathfrak{m}_A)$. Since, the derivations a,b are uniquely determined by the values $a(x_i),b(x_i)$, we will solve the equations $a(x_i)=(\rho b+b\rho)(x_i)$ recursively by induction on $-\deg(x_i)$. If x_i has degree 0, then $\rho(x_i)=0$ and then $a(x_i)\in\rho(R^{-1}\otimes\mathfrak{m}_A)$ (since a is trivial in cohomology): therefore, we can choose $b(x_i)\in R^{-1}\otimes\mathfrak{m}_A$ such that $a(x_i)=\rho b(x_i)$. Next, assume that x_i has degree $k<0$ and that $b(x_j)$ is defined for every x_j of degree bigger than k. Then $y_i:=a(x_i)-b(\rho x_i)$ is defined and $\rho(y_i)=\rho a(x_i)-\rho b(\rho x_i)=a(\rho x_i)-(a-b\rho)(\rho x_i)=0$. Since $H^k(R\otimes A,\rho)=0$, there exists $b(x_i)\in R^{k-1}\otimes\mathfrak{m}_A$ such that $\rho b(x_i)=y_i$. $\square$

We point out that the fact that the cohomology of R is concentrated in degree 0 plays an essential role in the above proof: for instance, if we take R as the Koszul complex of $x^2,x^3\in\mathbb{K}[x]$ then the result of Lemma 4.4 fails to be true.

For any DGLA L, the previous constructions allow us to define the corresponding formal objects:

(1) the *Maurer-Cartan functor* $\operatorname{MC}_L\colon\mathbf{Art}\to\mathbf{Set}$ by setting [Ma09]:
$$\operatorname{MC}_L(A)=\operatorname{MC}(L\otimes\mathfrak{m}_A);$$
(2) the deformation functor $\operatorname{Def}_L:\mathbf{Art}\longrightarrow\mathbf{Set}$:
$$\operatorname{Def}_L(A)=\frac{\operatorname{MC}_L(A)}{\text{gauge}};$$
(3) the *formal reduced Deligne groupoid* $\overline{\operatorname{Del}}_L\colon\mathbf{Art}\to\mathbf{Grpd}$:
$$\overline{\operatorname{Del}}_L(A)=\overline{\operatorname{Del}}(L\otimes\mathfrak{m}_A).$$

Every morphism of DGLAs induces a morphism of formal reduced groupoids and, therefore, a morphism of the associated Maurer-Cartan and deformation functors. A basic result asserts that if L and M are quasi-isomorphic DGLAs, then the associated functor $\overline{\operatorname{Del}}_L$ and $\overline{\operatorname{Del}}_M$ are equivalent [GM88, Kon94, Ma09, Ye11].

Given a semicosimplicial groupoid
$$G^\Delta:\qquad G_0 \rightrightarrows G_1 \;\substack{\rightarrow\\[-1em]\rightarrow\\[-1em]\rightarrow}\; G_2 \;\substack{\rightarrow\\[-1em]\rightarrow\\[-1em]\rightarrow\\[-1em]\rightarrow}\;\cdots$$
its total space is the groupoid $\operatorname{tot}(G^\Delta)$ defined in the following way [Hin97, FMM10]:

(1) The objects of $\operatorname{tot}(G^\Delta)$ are the pairs (l,m) with l an object in G_0 and m a morphism in G_1 between $\partial_0 l$ and $\partial_1 l$. Moreover the three images of m via the maps ∂_i are the edges of a 2-simplex in the nerve of G_2, i.e.
$$(\partial_0 m)(\partial_1 m)^{-1}(\partial_2 m)=1 \text{ in } (G_2)_1(\partial_2\partial_0 l,\partial_2\partial_0 l).$$

(2) The morphisms between (l_0, m_0) and (l_1, m_1) are morphisms a in G_0 between l_0 and l_1 making the diagram

$$\begin{tikzcd} \partial_0 l_0 \arrow[r, "m_0"] \arrow[d, "\partial_0 a"'] & \partial_1 l_0 \arrow[d, "\partial_1 a"] \\ \partial_0 l_1 \arrow[r, "m_1"] & \partial_1 l_1 \end{tikzcd}$$

commutative in G_1.

Example 4.5. Let

$$G^{\Delta}: \qquad G_0 \rightrightarrows G_1 \mathrel{\substack{\rightarrow\\\rightarrow\\\rightarrow}} G_2 \mathrel{\substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow}} \cdots$$

be a semicosimplicial groupoid. Assume that for every i the natural map $G_i \to \pi_0(G_i)$ is an equivalence, i.e., every G_i is equivalent to a set. Then also $\operatorname{tot}(G^{\Delta})$ is equivalent to a set and, more precisely, to the equalizer of the diagram of sets

$$\pi_0(G_0) \rightrightarrows \pi_0(G_1).$$

The next theorem is one of the main results of [Hin97, FIM09].

Theorem 4.6. *Let $\mathfrak{g}^{\Delta}$ be a semicosimplicial DGLA, such that $H^j(\mathfrak{g}_i) = 0$ for every i and $j < 0$ and let*

$$\overline{\mathrm{Del}}_{\mathfrak{g}^{\Delta}}: \qquad \overline{\mathrm{Del}}_{\mathfrak{g}_0} \rightrightarrows \overline{\mathrm{Del}}_{\mathfrak{g}_1} \mathrel{\substack{\rightarrow\\\rightarrow\\\rightarrow}} \overline{\mathrm{Del}}_{\mathfrak{g}_2} \mathrel{\substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow}} \cdots$$

the associated semicosimplicial formal reduced Deligne groupoid. Then, there exists a natural isomorphism of functors $\mathbf{Art} \to \mathbf{Set}$

$$\mathrm{Def}_{\mathrm{TW}(\mathfrak{g}^{\Delta})} = \pi_0(\operatorname{tot}(\overline{\mathrm{Del}}_{\mathfrak{g}^{\Delta}})).$$

Proof. This is Theorem 7.6 of [FIM09], expressed in terms of Deligne groupoids and irrelevant stabilizers. The same result is proved in [Hin97] under the assumption that every $\mathfrak{g}_i$ is concentrated in non negative degrees (and then Deligne=reduced Deligne). $\square$

5. Embedded deformations of complete intersection

Let X be a smooth algebraic variety over an algebraically closed field $\mathbb{K}$ of characteristic zero, and $Z \subset X$ a closed subvariety of pure codimension p as in the set-up of the introduction: there exist a Zariski open subset $U \subset X$, a locally free sheaf $\mathcal{E}$, of rank p over U, and a section $f \in \Gamma(U, \mathcal{E})$ such that $Z = \{f = 0\} \subset U$.

The aim of this section is to describe two convenient differential graded Lie algebras controlling the functor $\mathrm{Hilb}_{Z|X}\colon \mathbf{Art} \to \mathbf{Set}$ of infinitesimal embedded deformations of Z in X.

The first DGLA is simpler and it has a clear geometric interpretation. The second, which is quasi-isomorphic to the first one, will be used in the proof of our main result and it is similar to the ones considered in [Ma07, Ia10] in the case Z smooth.

5.1. **Local case.** Assume that $U = \operatorname{Spec} P$ is a smooth affine over $\mathbb{K}$ and $\mathcal{E} = \mathcal{O}_U^p$. If $f_1, \ldots, f_p$ are the components of the section f, then the ideal of Z is $J = (f_1, \ldots, f_p) \subset P$.

It is well known (see e.g. [Ar76, Se06]) that the set of embedded deformations of Z over a local Artin ring A corresponds naturally to the set of ideals $\tilde{J} \subset P \otimes A$ generated by liftings of $f_1, \ldots, f_p$. Notice that every lifting of f_i may be written as $f_i + g_i$ with $g_i \in P \otimes \mathfrak{m}_A$.

Let R be the Koszul complex of the sequence $f_1, \ldots, f_p$, considered as a DG-algebra; in other words, R is the polynomial algebra $P[y_1, \ldots, y_p]$, where $\deg(y_i) = -1$ and $d(y_i) = f_i$. In particular, $R^0 = P$, $R^{-1} = \bigoplus_i P y_i$ and $R^j = 0$ for every $j > 0$. Since Z is a complete intersection, the natural map $R \to P/J$ is a quasi-isomorphism of DG-algebras.

Lemma 5.1. *In the notation above, the differential graded Lie algebra $L = \operatorname{Der}^*_P(R,R)$ controls the embedded deformations of $Z \subset X$. More precisely, there exists an equivalence of formal groupoids*

$$\overline{\operatorname{Del}}_L \to \operatorname{Hilb}_{Z|X}$$

and, therefore, an isomorphism of functors of Artin rings

$$\operatorname{Def}_L \to \operatorname{Hilb}_{Z|X}\,.$$

Proof. Let A be a local Artin ring. We have $\operatorname{MC}_{\operatorname{Der}^*_P(R,R)}(A) = \operatorname{Der}^1_P(R,R) \otimes \mathfrak{m}_A = (P \otimes \mathfrak{m}_A)^p$, since every derivation η of R of degree 1 is uniquely determined by the sequence $\eta(y_1), \ldots, \eta(y_p) \in P \otimes \mathfrak{m}_A$. Then Example 4.3 gives a morphism of groupoids

$$\overline{\operatorname{Del}}_L(A) \to \operatorname{Hilb}_{Z|X}(A)$$

which is surjective on objects.

Let $\eta, \mu \in \operatorname{Der}^1_P(R,R) \otimes \mathfrak{m}_A$ be two derivations giving the same deformation, i.e.,

$$\tilde{J} = (f_1 + \eta(y_1), \ldots, f_p + \eta(y_p)) = (f_1 + \mu(y_1), \ldots, f_p + \mu(y_p))$$

as ideal of $P \otimes A$. Then, by the flatness of $\tilde{J}$, we have

$$\eta(y_i) - \mu(y_i) \in \ker(\tilde{J} \to \tilde{J} \otimes_A \mathbb{K} = J) = \mathfrak{m}_A \tilde{J}$$

and then there exist $a_{i1}, \ldots, a_{ip} \in P \otimes \mathfrak{m}_A$ such that

$$\eta(y_i) - \mu(y_i) = \sum_j a_{ij}(f_j + \mu(y_j)).$$

Taking $c \in \operatorname{Der}^0_P(R,R) \otimes \mathfrak{m}_A$ such that $e^c(y_i) = y_i + \sum_j a_{ij} y_j$ we have that e^c is an isomorphism of the two DG-algebras $(R \otimes A, d + \eta)$ and $(R \otimes A, d + \mu)$, i.e., $e^c * \eta = \mu$ and η, μ are gauge equivalent. The last step of the proof is exactly Lemma 4.4. ∎

As regard the second DGLA, consider the DG-algebra

$$S = P[z_1, \ldots, z_p, y_1, \ldots, y_p],$$

where $\deg z_i = 0$, $\deg y_i = -1$ and $d(y_i) = f_i - z_i$. Notice that the map

$$S \to P, \qquad y_i \mapsto 0, \quad z_i \mapsto f_i \tag{5.1}$$

is a surjective quasi-isomorphism of DG-algebras.

Let $I \subset S$ be the ideal generated by $z_1, \ldots, z_p$, i.e., the kernel of the projection map $\pi\colon S \to R$. Then, consider the DGLA $H = \{\eta \in \operatorname{Der}^*_P(S,S) \mid \eta(I) \subset I\}$ and the surjective morphism of DGLAs

$$\Phi\colon H \to \operatorname{Der}^*_P(R,R), \qquad \Phi(\eta)(y_i) = \pi(\eta(y_i)).$$

Lemma 5.2. *Let $\chi\colon H \to \operatorname{Der}^*_P(S,S)$ be the inclusion. Then, the horizontal maps*

$$\begin{tikzcd} H \arrow[r, "\Phi"] \arrow[d, "\chi"'] & \operatorname{Der}^*_P(R,R) \arrow[d] \\ \operatorname{Der}^*_P(S,S) \arrow[r] & 0 \end{tikzcd}$$

induce a quasi-isomorphism of homotopy fibers.

Proof. The horizontal maps are surjective and then it is sufficient to prove that

$$\operatorname{Der}^*_P(S,S) = \operatorname{Hom}^*_S(\Omega_{S/P}, S), \qquad \ker\Phi = \operatorname{Der}^*_P(S,I) = \operatorname{Hom}^*_S(\Omega_{S/P}, I)$$

are acyclic; both equalities follow from the fact that the S-module $\Omega_{S/P}$ is semifree and acyclic. ∎

5.2. **Global case.** The section $f\in\Gamma(U,\mathcal{E})$ gives a Koszul-Tate resolution of the structure sheaf of Z:

$$0\to\bigwedge^{p}\mathcal{E}^\vee\to\bigwedge^{p-1}\mathcal{E}^\vee\to\cdots\to\bigwedge^{2}\mathcal{E}^\vee\to\mathcal{E}^\vee\to\mathcal{O}_U\to\mathcal{O}_Z\to 0$$

and, therefore, a quasi-isomorphism of sheaves of DG-algebras over $\mathcal{O}_U$

$$\mathcal{R}=\operatorname{Sym}^*_{\mathcal{O}_U}(\mathcal{E}^\vee[1])\to\mathcal{O}_Z,$$

where the differential on $\mathcal{R}$ is given by extending $f^*\colon\mathcal{E}^\vee\to\mathcal{O}_U$ via Leibniz rule.

Let $\mathcal{D}er^*_{\mathcal{O}_U}(\mathcal{R},\mathcal{R})$ be the DG-sheaf of $\mathcal{O}_U$-linear derivations of $\mathcal{R}$, it is a sheaf of DGLAs and consider an affine open covering $\mathcal{U}=\{U_i\}$, which is trivializing for the sheaf $\mathcal{E}$.

Theorem 5.3. *In the above set-up, assume that $\mathcal{U}$ is an affine open cover of X which is trivializing for the locally free sheaf $\mathcal{E}$, then the infinitesimal embedded deformations of Z in X are controlled by the differential graded Lie algebra $TW(\mathcal{U},\mathcal{D}er^*_{\mathcal{O}_U}(\mathcal{R},\mathcal{R}))$.*

Proof. By the computations in the local case, for every i the DGLA $\Gamma(U_i,\mathcal{D}er^*_{\mathcal{O}_U}(\mathcal{R},\mathcal{R}))$ controls the embedded deformations of $Z\cap U_i$ inside U_i. A global embedded deformation of Z in X is simply given by a sequence of embedded deformations of $Z\cap U_i$ inside U_i, with isomorphic restrictions on double intersections U_{ij}. Therefore, to conclude the proof, it is enough to apply Theorem 4.6 and Example 4.5. □

In order to globalize the second local construction, let us consider the graded locally free $\mathcal{O}_U$-module $\mathcal{E}^\vee[1]\oplus\mathcal{E}^\vee$. On the graded symmetric $\mathcal{O}_U$-algebra $\operatorname{Sym}^*_{\mathcal{O}_U}(\mathcal{E}^\vee[1]\oplus\mathcal{E}^\vee)$, we consider the unique differential induced by the map of degree $+1$

$$\mathcal{E}^\vee[1]\oplus\mathcal{E}^\vee\to\operatorname{Sym}^*_{\mathcal{O}_U}(\mathcal{E}^\vee[1]\oplus\mathcal{E}^\vee),\qquad \mathcal{E}^\vee[1]\ni x\mapsto f^*(x)-x\in\mathcal{O}_U\oplus\mathcal{E}^\vee.\tag{5.2}$$

Let $\mathcal{M}=\mathcal{D}er^*_{\mathcal{O}_U}(\operatorname{Sym}^*_{\mathcal{O}_U}(\mathcal{E}^\vee[1]\oplus\mathcal{E}^\vee),\operatorname{Sym}^*_{\mathcal{O}_U}(\mathcal{E}^\vee[1]\oplus\mathcal{E}^\vee))$ and $\mathcal{M}_\perp\subset\mathcal{M}$ the subsheaf of derivations preserving the ideal generated by $\mathcal{E}^\vee$. Then, the homotopy fiber of the inclusion $\mathcal{M}_\perp\to\mathcal{M}$ is quasi-isomorphic to $\mathcal{D}er^*_{\mathcal{O}_U}(\mathcal{R},\mathcal{R})$ and we have the following result.

Theorem 5.4. *In the above set-up, assume that $\mathcal{U}$ is an affine open cover of U, then the infinitesimal embedded deformations of Z in X are controlled by the homotopy fiber of the inclusion of DGLAs*

$$TW(\mathcal{U},\mathcal{M}_\perp)\xrightarrow{\chi}TW(\mathcal{U},\mathcal{M}).$$

Proof. This is the global version of Lemma 5.2, as Theorem 5.3 is the global version of Lemma 5.1. □

5.3. **Relation with DG-schemes.** According to [CK01], a *DG-scheme* is a pair $(T,\mathcal{R}_T)$, where T is an ordinary scheme and $\mathcal{R}_T$ is a sheaf of ($\mathbb{Z}_{\le 0}$-graded) commutative DG-algebras on T, such that $\mathcal{R}^0_T=\mathcal{O}_T$ and each $\mathcal{R}^i_T$ is quasi-coherent over $\mathcal{O}_T$. A *morphism of DG-schemes* is just a morphism of DG-ringed spaces.

A *closed embedding of DG-schemes* is a morphism $f\colon(Y,\mathcal{R}_Y)\to(T,\mathcal{R}_T)$ such that $f\colon Y\to T$ is a closed embedding of schemes and the induced map $\mathcal{R}_T\to f_*\mathcal{R}_Y$ is surjective.

Any ordinary scheme can be be considered as a DG-scheme with trivial grading and differential; any ordinary closed subscheme of T can be considered as a closed DG-subscheme of $(T,\mathcal{R}_T)$.

For any DG-scheme $(T,\mathcal{R}_T)$, its differential $\delta\colon\mathcal{R}^i_T\to\mathcal{R}^{i+1}_T$ is $\mathcal{O}_T$-linear and hence $\mathcal{H}^i(\mathcal{R}_T)$ are quasi-coherent sheaves on T. We define *the degree 0 truncation* $\pi_0(T,\mathcal{R}_T)$ as the ordinary closed subscheme of T defined by the ideal $\delta(\mathcal{R}^{-1}_T)$, and then with structure sheaf $\mathcal{H}^0(\mathcal{R}_T)$.

A *quasi-isomorphism of DG-schemes* is a morphism $f\colon (Y,\mathcal{R}_Y)\to(T,\mathcal{R}_T)$ such that the induced map $\pi_0(Y,\mathcal{R}_Y)\to\pi_0(T,\mathcal{R}_T)$ is a isomorphism of schemes and $\mathcal{H}^*(\mathcal{R}_T)\to f_*\mathcal{H}^*(\mathcal{R}_Y)$ is an isomorphism of graded sheaves.

It is useful to see the constructions of Subsection 5.2 in the framework of DG-schemes; more precisely, we will describe two DG-schemes which are quasi-isomorphic to Z and U, respectively.

The first DG-scheme is just the pair $(U,\mathcal{R})$, defined in the previous subsection; in this case, we have $\pi_0(U,\mathcal{R})=Z$ and the closed embedding $(Z,\mathcal{O}_Z)\to(U,\mathcal{R})$ is a quasi-isomorphism of DG-schemes.

The other DG-scheme is $(E,\mathcal{S})$, where $\pi\colon E\to U$ is the total space of the vector bundle associated with $\mathcal{E}$ (i.e., $E=\operatorname{Spec}(\operatorname{Sym}^*_{\mathcal{O}_U}(\mathcal{E}^\vee))$ and $\mathcal{S}=\operatorname{Sym}^*_{\mathcal{O}_E}(\pi^*\mathcal{E}^\vee[1])$, with the differential on $\mathcal{S}$ induced by the morphism as in Equation (5.2). Notice that $\pi_*\mathcal{S}=\operatorname{Sym}^*_{\mathcal{O}_U}(\mathcal{E}^\vee[1]\oplus\mathcal{E}^\vee)$ and, by (5.1), the section $f\colon U\to E$ gives a closed embedding and a quasi-isomorphism $(U,\mathcal{O}_U)\to(E,\mathcal{S})$, while the zero section $0\colon U\to E$ induces a closed embedding $(U,\mathcal{R})\to(E,\mathcal{S})$.

Therefore, we have the following commutative diagram of closed embeddings of DG-schemes

$$\begin{tikzcd}
(Z,\mathcal{O}_Z) \arrow[r, "qiso"] \arrow[d] & (U,\mathcal{R}) \arrow[d] \\
(U,\mathcal{O}_U) \arrow[r, "qiso"] & (E,\mathcal{S})
\end{tikzcd}$$

Notice that we have the following natural isomorphisms of DG-sheaves

$$\mathcal{M}=\mathcal{D}er^*_{\mathcal{O}_U}(\pi_*\mathcal{S},\pi_*\mathcal{S})=\pi_*\,\mathcal{D}er^*_{\mathcal{O}_U}(\mathcal{S},\mathcal{S}),\qquad \mathcal{M}_\perp=\pi_*\{\phi\in\mathcal{D}er^*_{\mathcal{O}_U}(\mathcal{S},\mathcal{S})\mid\phi(\mathcal{I})\subset\mathcal{I}\},$$

where $\mathcal{I}$ is the ideal sheaf of the closed embedding $(U,\mathcal{R})\to(E,\mathcal{S})$.

6. L_∞ morphisms and obstructions

We briefly recall the notion of L_∞-algebras and L_∞ morphisms. For a more detailed description of such structures we refer to [SS79, LS93, LM95, Ma02, Fu03, Kon03, Get04, Ma04a, FiMa07] and [Ma04b, Chapter IX].

Definition 6.1. An L_∞ structure on a graded vector space V is a sequence $\{q_k\}_{k\geq1}$ of linear maps $q_k\in\operatorname{Hom}^1_{\mathbb{K}}(\bigodot^k(V[1]),V[1])$ such that the coderivation

$$Q:\overline{\bigodot}^*(V[1])\to\overline{\bigodot}^*(V[1]),$$

defined as

$$Q(v_1\odot\cdots\odot v_n)=\sum_{k=1}^{n}\sum_{\sigma\in S(k,n-k)}\epsilon(\sigma)q_k(v_{\sigma(1)}\odot\cdots\odot v_{\sigma(k)})\odot v_{\sigma(k+1)}\odot\cdots\odot v_{\sigma(n)},$$

satisfies $QQ=0$, i.e., if Q is a codifferential on the reduced symmetric graded coalgebra; here $\epsilon(\sigma)$ denotes the Koszul sign and $S(k,n-k)$ is the set of *unshuffles* of type $(k,n-k)$, i.e., the set of permutations σ such that $\sigma(1)<\sigma(2)<\cdots<\sigma(k)$ and $\sigma(k+1)<\sigma(k+2)<\cdots<\sigma(n)$.

An L_∞-algebra $(V,q_1,q_2,q_3,\ldots)$ is the data of a graded vector space V and an L_∞ structure $\{q_i\}$ on it.

Notice that if $(V,q_1,q_2,q_3,\ldots)$ is an L_∞-algebra, then $q_1q_1=0$ and therefore $(V[1],q_1)$ is a DG-vector space.

Example 6.2 (Quillen construction, [Qui69])**.** Let $(L, d, [\ ,\ ])$ be a differential graded Lie algebra and define:

$$q_1 = -d : L[1] \to L[1],$$
$$q_2 \in \operatorname{Hom}^1_{\mathbb{K}}(\bigodot^2 (L[1]), L[1]), \qquad q_2(v \odot w) = (-1)^{\overline{v}}[v, w],$$

and $q_k = 0$ for every $k \geq 3$. Then, $(L, q_1, q_2, 0, \ldots)$ is an L_∞-algebra. Explicitly, in this case, the differential Q is given by

$$Q(v_1 \odot \cdots \odot v_n) = \sum_{\sigma \in S(1,n-1)} \epsilon(\sigma) d(v_{\sigma(1)}) \odot \cdots \odot v_{\sigma(n)} + \sum_{\sigma \in S(2,n-2)} \epsilon(\sigma) q_2(v_{\sigma(1)} \odot v_{\sigma(2)}) \odot v_{\sigma(3)} \odot \cdots \odot v_{\sigma(n)}$$

There exist two equally important notions of morphisms of L_∞ structures: the linear morphisms (graphically denoted with a solid arrow) and the L_∞ morphisms (denoted with a dashed arrow).

A *linear morphism* $f\colon (V, q_1, q_2, \ldots) \to (W, p_1, p_2, \ldots)$ of L_∞-algebras is a morphism of graded vector spaces $f\colon V[1] \to W[1]$ such that $f \circ q_n = p_n \circ (\odot^n f)$, for every $n > 0$.

An L_∞ *morphism* $f_\infty\colon (V, q_1, q_2, \ldots) \dashrightarrow (W, p_1, p_2, \ldots)$ of L_∞-algebras is the data of a sequence of morphisms

$$f_n \in \operatorname{Hom}^0_{\mathbb{K}}(\bigodot^n (V[1]), W[1]), \qquad n \geq 1,$$

such that the unique morphism of graded coalgebras

$$F\colon \overline{\bigodot}^* (V[1]) \to \overline{\bigodot}^* W[1]$$

lifting $\sum_n f_n : \overline{\bigodot}^* (V[1]) \to W[1]$, commutes with the codifferentials. This condition implies that the *linear part* $f_1\colon V[1] \to W[1]$ of an L_∞ morphism $f_\infty\colon (V, q_1, q_2, \ldots) \dashrightarrow (W, p_1, p_2, \ldots)$ satisfies the condition $f_1 \circ q_1 = p_1 \circ f_1$, and therefore $f_1\colon (V[1], q_1) \to (W[1], p_1)$ is a morphism of DG-vector spaces.

Remark 6.3. Every linear morphism of L_∞-algebras is also an L_∞ morphism, conversely an L_∞ morphism $f_\infty = \{f_n\}$ is linear if and only if $f_n = 0$, for every $n \geq 2$. An L_∞ morphism between two DGLAs is linear if and only if it is a morphism of differential graded Lie algebras.

Lemma 6.4. *Let L and M be differential graded Lie algebras, $f_\infty\colon L \dashrightarrow M$ an L_∞ morphism between L and M and A a commutative DG-algebra. Then, f_∞ induces canonically an L_∞ morphism*

$$(f_A)_\infty\colon L \otimes A \dashrightarrow M \otimes A.$$

Proof. By definition, $f_\infty\colon L \dashrightarrow M$ corresponds to a sequence $\{f_n\}_n$ of degree zero linear maps, with $f_n : \bigodot^n (L[1]) \to M[1]$. Then, for any associative graded commutative algebra A, define the natural extensions

$$(f_A)_n\colon \bigodot^n ((L \otimes A)[1]) \to (M \otimes A)[1]$$

as the composition

$$\bigodot^n ((L \otimes A)[1]) \xrightarrow{\mu} \bigodot^n (L[1]) \otimes A \xrightarrow{f_n \otimes Id_A} M[1] \otimes A \simeq (M \otimes A)[1],$$

where μ is induced by multiplication on A and by the Koszul rule of signs. The proof that this induces an L_∞ morphism is completely straightforward. □

Proposition 6.5. *Let* **D** *be a small category,* **DGLA** *the category of differential graded Lie algebras and $H, G\colon$* **D** $\to$ **DGLA** *two functors; denote by* $\lim(H)$ *and* $\lim(G)$ *their limits. Then, every L_∞ natural transformation $H \dashrightarrow G$ induces an L_∞ morphism*

$$f_\infty\colon \lim(H) \dashrightarrow \lim(G).$$

Proof. An L_∞ natural transformation is just a sequence

$$(f_d)_\infty \colon H(d) \dashrightarrow G(d), \qquad d \in D,$$

of L_∞ morphisms, such that for every morphism $\gamma\colon d \to d'$ in D the diagram

$$\begin{tikzcd} H(d) \arrow[r, dashed, "(f_d)_\infty"] \arrow[d, "H(\gamma)"] & G(d) \arrow[d, "G(\gamma)"] \\ H(d') \arrow[r, dashed, "(f_{d'})_\infty"] & G(d') \end{tikzcd}$$

is commutative in the category of L_∞-algebras. Then, the conclusion follows immediately from the fact that the Quillen's construction, considered as a functor from the category of DGLAs to the category of locally nilpotent differential graded coalgebras, commutes with limits. Notice that the forgetful functor from coalgebras to vector spaces does not commute with limits. $\square$

Corollary 6.6. *Let $\mathfrak{g}^\Delta$ and $\mathfrak{h}^\Delta$ be two semicosimplicial DGLAs. Then every semicosimplicial L_∞ morphism $\mathfrak{g}^\Delta \dashrightarrow \mathfrak{h}^\Delta$ induces an L_∞ morphism*

$$TW(\mathfrak{g}^\Delta) \dashrightarrow TW(\mathfrak{h}^\Delta).$$

Proof. By definition, a semicosimplicial L_∞ morphism is a sequence of L_∞ morphisms $\{(f_d)_\infty : \mathfrak{g}_d \dashrightarrow \mathfrak{h}_d\}$, $d \geq 0$, commuting with coface maps. By definition, we have

$$TW(\mathfrak{g}^\Delta) = \{(x_n) \in \prod_n (A_{PL})_n \otimes \mathfrak{g}_n \mid (\delta^k \otimes Id)x_n = (Id \otimes \partial_k)x_{n-1} \;\forall n,k\}.$$

Then, applying Lemma 6.4, every $(f_n)_\infty$ induces L_∞ morphisms $(A_{PL})_m \otimes \mathfrak{g}_n \dashrightarrow (A_{PL})_m \otimes \mathfrak{h}_n$. Finally, it is enough to apply the previous Proposition 6.5 interpreting TW as an end, and then as a limit. $\square$

Example 6.7. Let $\chi\colon L \to M$ and $\eta\colon L' \to M'$ be morphisms of differential graded Lie algebras. Then, every commutative diagram

$$\begin{tikzcd} L \arrow[r, dashed, "f"] \arrow[d, "\chi"] & L' \arrow[d, "\eta"] \\ M \arrow[r, dashed, "f'"] & M' \end{tikzcd}$$

with f and f' L_∞ morphisms, induces an L_∞ morphism $TW(\chi) \dashrightarrow TW(\eta)$.

Remark 6.8. Via the standard décalage isomorphisms $\bigodot^n(V[1]) \simeq (\bigwedge^n V)[n]$, every L_∞ morphism $f_\infty\colon V \dashrightarrow W$, associated with a sequence of maps $f_n : \bigodot^n(V[1]) \to W[1]$ can be described by a sequence of linear maps

$$g_n\colon \bigwedge^n V \to W, \qquad n \geq 1, \quad \deg(g_n) = 1-n.$$

Every L_∞-morphism $f_\infty\colon L \dashrightarrow M$ of DGLAs induces morphisms of the associated Maurer-Cartan and deformation functors

$$f_\infty\colon \operatorname{MC}_L \to \operatorname{MC}_M, \qquad f_\infty\colon \operatorname{Def}_L \to \operatorname{Def}_M.$$

Moreover, the linear part $f_1\colon L[1] \to M[1]$ induces a morphism in cohomology, which is compatible with the obstruction maps of Def_L and Def_M in the following sense: given a DGLA L, a small extension in the category **Art**

$$e: \quad 0 \to \mathbb{K} \to B \to A \to 0, \qquad A, B \in \mathbf{Art},$$

and an element $x \in \operatorname{MC}_L(A)$ we can take a lifting $\tilde{x} \in L^1 \otimes \mathfrak{m}_B$ and consider the element $h = d\tilde{x} + [\tilde{x}, \tilde{x}]/2 \in L^2 \otimes \mathbb{K} = L^2$. It is very easy to show that $dh = 0$ and that the

cohomology class $[h] \in H^2(L)$ does not depend on the choice of the lifting; therefore, it gives the *obstruction map* $ob_e\colon \operatorname{MC}_L(A) \to H^2(L)$. It is not difficult to prove (see e.g. [FaMa98, Ma09]) that the obstruction map is gauge invariant, thus giving a map $ob_e\colon \operatorname{Def}_L(A) \to H^2(L)$ such that an element $x \in \operatorname{Def}_L(A)$ lifts to $\operatorname{Def}_L(B)$ if and only if $ob_e(x) = 0$. The obstructions of the functor Def_L are the elements of $H^2(L)$ lying in the image of some obstruction map; for instance, if L is abelian then every obstruction is equal to 0. The obstruction maps commute with morphisms of DGLAs and quasi-isomorphic DGLAs have isomorphic associated deformation functors and the same obstructions. This implies that the set of obstructions is a homotopical invariant of the DGLA.

Given an L_∞ morphism $L \dashrightarrow M$ of DGLAs, the map $H^2(L) \to H^2(M)$, induced by its linear part, commutes with the induced morphism $\operatorname{Def}_L \to \operatorname{Def}_M$ and obstruction maps [Ma02]. In particular, if M is homotopy abelian, then the obstructions of the functor Def_L are contained in the kernel of the induced map $H^2(L) \to H^2(M)$.

The obstruction maps exist for every functor of Artin rings $F\colon \mathbf{Art} \to \mathbf{Set}$ describing infinitesimal deformation of algebro-geometric structures and one of the main results of [FaMa98] is the proof that obstruction maps exist for every F as above satisfying some mild Schlessinger type conditions.

Example 6.9. Let $\chi\colon L \to M$ be a morphism of DGLAs, then the obstructions of the functor $\operatorname{Def}_{TW(\chi)}\colon \mathbf{Art} \to \mathbf{Set}$ are contained in $H^2(TW(\chi))$. If χ is injective, then $H^2(TW(\chi)) \cong H^1(\operatorname{coker}(\chi))$ (see Remark 2.2). Moreover, if χ is also injective in cohomology then $TW(\chi)$ is homotopy abelian (Lemma 2.1) and then $\operatorname{Def}_{TW(\chi)}$ has only trivial obstructions.

Example 6.10. Let X be a smooth algebraic variety, over an algebraically closed field $\mathbb{K}$ of characteristic 0, and $Z \subset X$ a locally complete intersection closed subvariety. Let $\operatorname{Hilb}_{Z|X}\colon \mathbf{Art} \to \mathbf{Set}$ the deformation functor of infinitesimal embedded deformations of Z in X. Then, it is well known that the obstructions are naturally contained in the cohomology vector space $H^1(Z, N_{Z|X})$ of the normal sheaf of Z in X, see e.g [Se06, Prop. 3.2.6]. This is also recovered in this paper as a consequence of Theorem 5.3.

7. Cartan homotopies and L_∞ morphisms

Next, let us recall the notion of Cartan homotopy [FiMa07, FiMa09].

Definition 7.1. Let L and M be two differential graded Lie algebras. A linear map of degree -1

$$\boldsymbol{i}\colon L \to M$$

is called a *Cartan homotopy* if, for every $a, b \in L$, we have

$$[\boldsymbol{i}_a, \boldsymbol{i}_b] = 0, \qquad \boldsymbol{i}_{[a,b]} = [\boldsymbol{i}_a, d_M \boldsymbol{i}_b].$$

Lemma 7.2. *Let $\boldsymbol{i}\colon L \to M$ be a Cartan homotopy and consider the degree 0 map*

$$\boldsymbol{l}\colon L \to M, \qquad \boldsymbol{l}_a = d_M \boldsymbol{i}_a + \boldsymbol{i}_{d_L a}.$$

Then:

1. *$\boldsymbol{l}$ is a morphism of DGLAs;*
2. *$\boldsymbol{i}_{[a,b]} = [\boldsymbol{i}_a, \boldsymbol{l}_b] = (-1)^{\overline{a}}[\boldsymbol{l}_a, \boldsymbol{i}_b]$;*
3. $[\boldsymbol{i}_{[a,b]}, \boldsymbol{l}_c] - (-1)^{\overline{b}\,\overline{c}}[\boldsymbol{i}_{[a,c]}, \boldsymbol{l}_b] + (-1)^{\overline{a}(\overline{b}+\overline{c})}[\boldsymbol{i}_{[b,c]}, \boldsymbol{l}_a] = 0.$

Proof. The proof of the first two items is straightforward. For every a, b and c, we have

$$[\boldsymbol{i}_{[a,b]}, \boldsymbol{l}_c] - (-1)^{\overline{b}\,\overline{c}}[\boldsymbol{i}_{[a,c]}, \boldsymbol{l}_b] + (-1)^{\overline{a}(\overline{b}+\overline{c})}[\boldsymbol{i}_{[b,c]}, \boldsymbol{l}_a]$$
$$= \boldsymbol{i}_{[[a,b],c]} - (-1)^{\overline{b}\,\overline{c}}\boldsymbol{i}_{[[a,c],b]} + (-1)^{\overline{a}(\overline{b}+\overline{c})}\boldsymbol{i}_{[[b,c],a]} = 0,$$

where the last equality follows from the graded Jacobi identity in L:

$$[[a, b], c] - (-1)^{\overline{b}\,\overline{c}}[[a, c], b] + (-1)^{\overline{a}(\overline{b}+\overline{c})}[[b, c], a] = 0. \tag{7.1}$$

Notice that the above form of Jacobi identity can be written as

$$\sum_{\sigma\in S(2,1)} \chi(\sigma)[[a_{\sigma(1)}, a_{\sigma(2)}], a_{\sigma(3)}] = 0,$$

where $\chi(\sigma)$ is the product of the signature and the Koszul sign of the unshuffle σ.

□

Lemma 7.3. *Let $= (L, d_L, [\,,\,])$ and $= (M, d_M, [\,,\,])$ be DGLAs. Consider the linear morphisms*

$$f_1 : L \to M \text{ and } f_2 : \bigwedge^2 L \to M,$$

where f_1 has degree zero and f_2 has degree -1 (see Remark 6.8). Then, the sequence $\{f_1, f_2, 0, 0, \cdots\}$ is an L_∞ morphism of DGLAs if and only if the following equations are satisfied for every $a, b, c, d \in L$.

1. $f_1 \circ d_L = d_M \circ f_1$ *(i.e., f_1 is a morphism of complexes);*
2. $[f_1(a), f_1(b)] = f_1([a, b]) - d_M \circ f_2(a, b) - f_2(d_La, b) + (-1)^{\overline{a}\overline{b}} f_2(d_Lb, a)$*;*
3. $f_2([a, b], c) - (-1)^{\overline{b}\overline{c}} f_2([a, c], b) + (-1)^{\overline{a}(\overline{b}+\overline{c})} f_2([b, c], a) =$
   $= (-1)^a[f_1(a), f_2(b, c)] - [f_2(a, b), f_1(c)] + (-1)^{\overline{b}\overline{c}}[f_2(a, c), f_1(b)]$*;*
4. $[f_2(a, b), f_2(c, d)] + (-1)^{(\overline{c}+1)(\overline{b}+1)}[f_2(a, c), f_2(b, d)] = 0.$

Proof. It follows from the explicit formula of L_∞ morphism of DGLAs, given for instance in [Ke05, LM95].

□

Theorem 7.4. *Let L, M be DGLAs and $\boldsymbol{i}\colon L \to M$ be a Cartan homotopy. Then, the sequence of linear maps*

$$g_1 \in \operatorname{Hom}^0(L, M[t, dt]), \qquad g_1(a) = t\boldsymbol{l}_a + dt\boldsymbol{i}_a,$$
$$g_2 \in \operatorname{Hom}^{-1}(\bigwedge^2 L, M[t, dt]), \qquad g_2(a, b) = t(1 - t)\boldsymbol{i}_{[a,b]},$$
$$g_n = 0 \quad \forall n \geq 3,$$

defines an L_∞ morphism $L \dashrightarrow M[t, dt]$ (see Remark 6.8).

Proof. We need to check the four conditions of Lemma 7.3. As regards (1), since $\boldsymbol{l}$ is a morphism of DGLAs and $d\boldsymbol{i} + \boldsymbol{i}d = \boldsymbol{l}$ we have

$$d_M(g_1(a)) = dt\boldsymbol{l}_a + td(\boldsymbol{l}_a) - dtd(\boldsymbol{i}_a) = dt\boldsymbol{l}_a + t\boldsymbol{l}_{da} - dt(\boldsymbol{l}_a - \boldsymbol{i}_{da}) = t\boldsymbol{l}_{da} + dt\boldsymbol{i}_{da} = g_1(d_L(a)).$$

As regards (2), we have

$$g_1([a, b]) - d_M \circ g_2(a, b) - g_2(d_La, b) + (-1)^{\overline{a}\overline{b}} g_2(d_Lb, a) =$$
$$t\boldsymbol{l}_{[a,b]} + dt\boldsymbol{i}_{[a,b]} - d_M(t(1 - t)\boldsymbol{i}_{[a,b]}) - t(1 - t)\boldsymbol{i}_{[da,b]} + (-1)^{\overline{a}\overline{b}} t(1 - t)\boldsymbol{i}_{[db,a]} =$$
$$t^2[\boldsymbol{l}_a, \boldsymbol{l}_b] + 2tdt\boldsymbol{i}_{[a,b]} = t^2[\boldsymbol{l}_a, \boldsymbol{l}_b] + tdt((-1)^{\overline{a}}[\boldsymbol{l}_a, \boldsymbol{i}_b] + [\boldsymbol{i}_a, \boldsymbol{l}_b])$$
$$= [g_1(a), g_1(b)]$$

since we have

$$-d_M(t(1-t)\boldsymbol{i}_{[a,b]}) = -((1-2t)dt\boldsymbol{i}_{[a,b]} + t(1-t)d(\boldsymbol{i}_{[a,b]})) =$$
$$(2t-1)dt\boldsymbol{i}_{[a,b]} + t(t-1)(\boldsymbol{l}_{[a,b]} + \boldsymbol{i}_{[da,b]} - (-1)^{\overline{a}\overline{b}}\boldsymbol{i}_{[db,a]}).$$

As regards Condition (3), we have that

$$g_2([a,b],c) - (-1)^{\overline{b}\overline{c}}g_2([a,c],b) + (-1)^{\overline{a}(\overline{b}+\overline{c})}g_2([b,c],a) =$$
$$(1-t)t(\boldsymbol{i}_{[[a,b],c]} - (-1)^{\overline{b}\overline{c}}\boldsymbol{i}_{[[a,c],b]} + (-1)^{\overline{a}(\overline{b}+\overline{c})}\boldsymbol{i}_{[[b,c],a]}) = 0$$

by Equation (7.1) of Lemma 7.2; moreover,

$$(-1)^a[g_1(a), g_2(b,c)] - [g_2(a,b), g_1(c)] + (-1)^{\overline{b}\overline{c}}[g_2(a,c), g_1(b)]$$
$$= t^2(1-t)((-1)^a[\boldsymbol{l}_a, \boldsymbol{i}_{[b,c]}] - [\boldsymbol{i}_{[a,b]}, \boldsymbol{l}_c] + (-1)^{\overline{b}\overline{c}}[\boldsymbol{i}_{[a,c]}, \boldsymbol{l}_b]) = 0$$

by Lemma 7.2. Finally, condition (4) follows from the fact that $[\boldsymbol{i}_x, \boldsymbol{i}_y] = 0$, for every choice of $x, y \in L$. □

Corollary 7.5. *In the same assumption of Theorem 7.4, let $N \subset M$ be a differential graded Lie subalgebra such that $\boldsymbol{l}(L) \subset N$ and*

$$TW(\chi) = \{(x, y(t)) \in N \times M[t, dt] \mid y(0) = 0,\ y(1) = x\}$$

the homotopy fiber of the inclusion $\chi\colon N \hookrightarrow M$. Then, the maps

$$g_1 \in \operatorname{Hom}^0(L, TW(\chi)), \qquad g_1(a) = (\boldsymbol{l}_a, t\boldsymbol{l}_a + dt\boldsymbol{i}_a),$$
$$g_2 \in \operatorname{Hom}^{-1}(\bigwedge^2 L, TW(\chi)), \qquad g_2(a,b) = (0, (1-t)t\boldsymbol{i}_{[a,b]}),$$
$$g_n = 0 \quad \forall n \geq 3,$$

define an L_∞ morphism $L \dashrightarrow TW(\chi)$.

Proof. Since $TW(\chi) \subset N \times M[t, dt]$, we need to check the 4 conditions of Lemma 7.3 on both components. As regard the first component, Conditions (1) and (2) follows from the fact that $\boldsymbol{l}$ is a morphism of DGLAs (and so it commutes with differentials and brackets). Conditions (3) and (4) are trivial, since $g_{2|N} = 0$. As regard the second components, it is due to the previous theorem.

□

The following definition is the natural modification of the definition of Tamarkin-Tsygan calculus [TT05], whenever the Gerstenhaber algebra is replaced by a DGLA.

Definition 7.6. Let L be a differential graded Lie algebra and V a differential graded vector space. A bilinear map

$$L \times V \xrightarrow{\lrcorner} V$$

of degree -1 is called a *calculus* if the induced map

$$\boldsymbol{i}\colon L \to \operatorname{Hom}^*_{\mathbb{K}}(V, V), \qquad \boldsymbol{i}_l(v) = l \lrcorner v,$$

is a Cartan homotopy.

The notion of calculus is stable under scalar extensions, more precisely, we have the following result.

Lemma 7.7. *Let V be a differential graded vector space and*

$$L \times V \xrightarrow{\lrcorner} V$$

a calculus. Then, for every differential graded commutative algebra A, the natural extension

$$(L \otimes A) \times (V \otimes A) \xrightarrow{\lrcorner} (V \otimes A) \qquad (l \otimes a) \lrcorner (v \otimes b) = (-1)^{\overline{a}\ \overline{v}} l \lrcorner v \otimes ab,$$

is a calculus.

Proof. Straightforward, see [IM10, Lemma 4.7]. □

The notions of Cartan homotopy and calculus extend naturally to the semicosimplicial objects and sheaves. Here, we consider only the case of calculus.

Definition 7.8. Let $\mathfrak{g}^\Delta$ be a semicosimplicial DGLA and V^Δ a semicosimplicial differential graded vector space. A semicosimplicial Lie-calculus

$$\mathfrak{g}^\Delta \times V^\Delta \xrightarrow{\lrcorner} V^\Delta,$$

is a sequence of calculi $\mathfrak{g}_n \times V_n \xrightarrow{\lrcorner} V_n$, $n \geq 0$, commuting with coface maps, i.e., $\partial_k(l \lrcorner v) = \partial_k(l) \lrcorner \partial_k(v)$, for every k.

Lemma 7.9. *Every semicosimplicial calculus*

$$\mathfrak{g}^\Delta \times V^\Delta \xrightarrow{\lrcorner} V^\Delta$$

extends naturally to a calculus

$$TW(\mathfrak{g}^\Delta) \times TW(V^\Delta) \xrightarrow{\lrcorner} TW(V^\Delta).$$

Proof. Straightforward, see [IM10, Proposition 4.9]. □

8. Calculus on de Rham complex of a DG-scheme

Let S be a DG-algebra and $d\colon S \to \Omega_{S/\mathbb{K}}$ its universal derivation. Denote by $\Omega^0_S = S$, $\Omega^1_S = \Omega_{S/\mathbb{K}}[-1]$ and

$$\Omega^k_S = \operatorname{Sym}^k_S \Omega^1_S.$$

The S-module Ω^1_S is generated by the elements da, $a \in S$ and $\deg(da) = \deg(a) + 1$. In the algebra $\Omega^*_S = \bigoplus_k \Omega^k_S$ we have

$$da \wedge db = (-1)^{(\overline{a}+1)(\overline{b}+1)} db \wedge da.$$

The $\mathbb{K}$-linear map of degree $+1$ $d\colon S \to \Omega^1_S$ extends to a unique differential (de Rham) of the graded commutative algebra Ω^*_S, i.e., $d\colon \Omega^*_S \to \Omega^*_S$. Notice that $d \in \operatorname{Der}^1_{\mathbb{K}}(\Omega^*_S, \Omega^*_S)$, $d^2 = 0$ and $d\Omega^i_S \subset \Omega^{i+1}_S$.

Next, assume that $\alpha \in \operatorname{Der}^k_{\mathbb{K}}(S, S)$ is a $\mathbb{K}$-linear derivation. Then α induces a morphism

$$\boldsymbol{i}_\alpha \in \operatorname{Hom}^{k-1}_S(\Omega^1_S, S) = \operatorname{Hom}^k_S(\Omega_{S/\mathbb{K}}, S)$$

such that $\boldsymbol{i}_\alpha(da) = \alpha(a)$, for every $a \in S$. By Leibniz rule, $\boldsymbol{i}_\alpha$ extends to a S-derivation of the graded symmetric algebra Ω^*_S, i.e.,

$$\boldsymbol{i}_\alpha \in \operatorname{Der}^{k-1}_S(\Omega^*_S, \Omega^*_S).$$

We shall call $\boldsymbol{i}_\alpha\colon \Omega^*_S \to \Omega^*_S$ the *interior product* by α. Notice that $\boldsymbol{i}_\alpha(\Omega^k_S) \subset \Omega^{k-1}_S$. Define the Lie derivation

$$L_\alpha := [\boldsymbol{i}_\alpha, d] \in \operatorname{Der}^k_{\mathbb{K}}(\Omega^*_S, \Omega^*_S).$$

Since L_α is $[-, d]$-exact it is also $[-, d]$-closed, i.e., $[L_\alpha, d] = 0$; if $a \in S = \Omega^0_S$ we have

$$L_\alpha(a) = \boldsymbol{i}_\alpha(da) = \alpha(a), \qquad L_\alpha(da) = (-1)^k d(L_\alpha(a)) = (-1)^k d(\alpha(a)).$$

Equivalently, L_α is the unique derivation extending $\alpha\colon \Omega^0_S \to \Omega^0_S$, such that $\deg(L_\alpha) = \deg(\alpha)$ and $[d, L_\alpha] = 0$. Notice that $L_\alpha(\Omega^i_S) \subset \Omega^i_S$.

If $\delta \in \operatorname{Der}^1_{\mathbb{K}}(S, S)$ is a differential, then also $L_\delta\colon \Omega^*_S \to \Omega^*_S$ is a differential: in fact, for every $a \in S$

$$L^2_\delta(a) = \delta^2(a) = 0, \qquad L^2_\delta(da) = L_\delta(-d\delta(a)) = d\delta^2(a) = 0.$$

In such case, $(\Omega^*_S, d, L_\delta)$ is a double complex and $(d + L_\delta)^2 = 0$.

We conclude the section, describing some properties of $\boldsymbol{i}_\alpha$ and L_α, for $\alpha \in \operatorname{Der}^k_{\mathbb{K}}(S, S)$.

Lemma 8.1. *Given $\alpha, \beta \in \operatorname{Der}^*_{\mathbb{K}}(S,S)$, we have the equalities:*
1. $[\boldsymbol{i}_\alpha, \boldsymbol{i}_\beta] = 0$;
2. $L_\beta = [\boldsymbol{i}_\beta, d] = (-1)^{\overline{\beta}}[d, \boldsymbol{i}_\beta]$;
3. $\boldsymbol{i}_{[\alpha,\beta]} = [L_\alpha, \boldsymbol{i}_\beta] = [[\boldsymbol{i}_\alpha, d], \boldsymbol{i}_\beta]$;
4. $[L_\alpha, d] = 0$, $L_{[\alpha,\beta]} = [L_\alpha, L_\beta]$;
5. $[L_\delta, \boldsymbol{i}_\beta] + \boldsymbol{i}_{[\delta,\beta]} = 0$, *for every* $\delta \in \operatorname{Der}^1_{\mathbb{K}}(S,S)$;
6. $\boldsymbol{i}_{[\alpha,\beta]} = [[\boldsymbol{i}_\alpha, d + L_\delta], \boldsymbol{i}_\beta] = [\boldsymbol{i}_\alpha, [[d + L_\delta], \boldsymbol{i}_\beta]]$, *for every* $\delta \in \operatorname{Der}^1_{\mathbb{K}}(S,S)$.

Proof. Since $\boldsymbol{i}_\alpha \boldsymbol{i}_\beta(da) = 0$, for every α, β, we have $[\boldsymbol{i}_\alpha, \boldsymbol{i}_\beta] = 0$.

Next, as regard (3), we have

$$\boldsymbol{i}_{[\alpha,\beta]}(da) = [\alpha,\beta]a = \alpha(\beta(a)) - (-1)^{\overline{\alpha}\,\overline{\beta}}\beta(\alpha(a)) = [L_\alpha, L_\beta]a$$

and

$$[L_\alpha, \boldsymbol{i}_\beta](da) = L_\alpha \boldsymbol{i}_\beta(da) - (-1)^{(\overline{\beta}-1)\overline{\alpha}} \boldsymbol{i}_\beta L_\alpha(da) = L_\alpha L_\beta(a) - (-1)^{\overline{\beta}\,\overline{\alpha}} \boldsymbol{i}_\beta d(L_\alpha a) = [L_\alpha, L_\beta]a.$$

As regard (4), we have

$$L_{[\alpha,\beta]} = [\boldsymbol{i}_{[\alpha,\beta]}, d] = [[L_\alpha, \boldsymbol{i}_\beta], d] = [L_\alpha, [\boldsymbol{i}_\beta, d]] = [L_\alpha, L_\beta].$$

As regard (5), for every $\delta \in \operatorname{Der}^1_{\mathbb{K}}(S,S)$, we have

$$\boldsymbol{i}_{[\delta,\beta]} = -(-1)^{\overline{\beta}} \boldsymbol{i}_{[\beta,\delta]} = (-1)^{\overline{\beta}}[\boldsymbol{i}_\beta, L_\delta] = -[L_\delta, \boldsymbol{i}_\beta].$$

The last one follows from

$$[[\boldsymbol{i}_\alpha, L_\delta], \boldsymbol{i}_\beta] = \pm[\boldsymbol{i}_{[\alpha,\delta]}, \boldsymbol{i}_\beta] = 0.$$

□

Proposition 8.2. *Let S be a DG-algebra and $\delta \in \operatorname{Der}^1_{\mathbb{K}}(S,S)$ a differential. Then, the map*

$$\boldsymbol{i} \colon (\operatorname{Der}^*_{\mathbb{K}}(S,S), [\delta, -]) \to (\operatorname{Der}^*_{\mathbb{K}}(\Omega^*_S, \Omega^*_S), [d + L_\delta, -]),$$

is a Cartan homotopy with $[d + L_\delta, \boldsymbol{i}_\alpha] + \boldsymbol{i}_{[\delta,\alpha]} = (-1)^{\overline{\alpha}} L_\alpha$.

Proof. By Lemma 8.1, we have

$$[d + L_\delta, \boldsymbol{i}_\alpha] + \boldsymbol{i}_{[\delta,\alpha]} = [d, \boldsymbol{i}_\alpha] = (-1)^{\overline{\alpha}} L_\alpha$$

and then

$$\boldsymbol{i}_{[\alpha,\beta]} = [\boldsymbol{i}_\alpha, [d, \boldsymbol{i}_\beta]] = [\boldsymbol{i}_\alpha, [d + L_\delta, \boldsymbol{i}_\beta] + \boldsymbol{i}_{[\delta,\beta]}] = [\boldsymbol{i}_\alpha, [d + L_\delta, \boldsymbol{i}_\beta]].$$

□

Assume that $S = \bigoplus_{i \le 0} S^i$ and let $T \subset S^0$ be a multiplicative subset. Then, for any S-DG-module M, every $\mathbb{K}$-derivation $S \to M$ extends in a unique way to a derivation $T^{-1}S \to M$; therefore, $\Omega_{T^{-1}S/\mathbb{K}} = T^{-1}\Omega_{S/\mathbb{K}}$,

$$\Omega^*_{T^{-1}S} = T^{-1}\Omega^*_S.$$

It is easy to verify that the two natural maps

$$\operatorname{Der}^*_{\mathbb{K}}(S,S) \to \operatorname{Der}^*_{\mathbb{K}}(T^{-1}S, T^{-1}S), \qquad \operatorname{Der}^*_{\mathbb{K}}(\Omega^*_S, \Omega^*_S) \to \operatorname{Der}^*_{\mathbb{K}}(\Omega^*_{T^{-1}S}, \Omega^*_{T^{-1}S}),$$

commute with the de Rham differential, interior products, Lie derivative and Cartan formulas, giving therefore a canonical morphism of calculi

$$\begin{tikzcd}
\operatorname{Der}^*_{\mathbb{K}}(S,S) \times \Omega^*_S \arrow[r] \arrow[d] & \Omega^*_S \arrow[d] \\
\operatorname{Der}^*_{\mathbb{K}}(T^{-1}S, T^{-1}S) \times \Omega^*_{T^{-1}S} \arrow[r] & \Omega^*_{T^{-1}S}
\end{tikzcd}$$

This implies that, for every DG-scheme $(E,\mathcal{S}_E)$ over the field $\mathbb{K}$, the interior product induces a morphism of sheaves

$$Der^*_{\mathbb{K}}(\mathcal{S}_E,\mathcal{S}_E)\times\Omega^*_{\mathcal{S}_E}\to\Omega^*_{\mathcal{S}_E}$$

which is still a calculus, where $\Omega^*_{\mathcal{S}_E}$ is the exterior algebra of the quasi-coherent sheaf of differentials on E.

9. Local cohomology of DG-sheaves

The theory of local cohomology extends naturally to DG-sheaves over DG-schemes. In order to fix the notation and prove some homotopy invariant properties, we briefly recall its construction in the quasi-coherent case, using the approach of Čech complex [BS98].

Let $A=\bigoplus_i A^i$ be a fixed DG-ring. Given an element $a\in Z^0(A)$ and an A-module $M=\oplus_i M^i$, we will denote $M_a=\oplus_i M^i_a$ the localization of M by the multiplicative subset of powers of a.

Since the localization is an exact functor on the category of $Z^0(A)$-modules, we have $H^i(M_a)=H^i(M)_a$, for every $i\in\mathbb{Z}$.

Let $\underline{a}=(a_1,\dots,a_n)$ be a sequence of elements of $Z^0(A)$; for every A-module M and every subset $H=\{h_1<h_2<\cdots<h_j\}\subseteq\{1,\dots,n\}$, denote by

$$a_\emptyset=1,\quad a_H=a_{h_1}\cdots a_{h_j}.$$

Notice that $a_{H\cup K}|a_Ha_K|a^2_{H\cup K}$, for every $H,K\subseteq\{1,\dots,n\}$; therefore, $(M_{a_H})_{a_K}=M_{a_{H\cup K}}$ and if $H\subset K$ there exists a natural localization map $M_{a_H}\to M_{a_K}$.

Definition 9.1. In the above situation, denote by $\check{C}^*(a_1,\dots,a_n;M)$ the complex of A-modules

$$0\to\check{C}^0(a_1,\dots,a_n;M)\xrightarrow{\delta}\check{C}^1(a_1,\dots,a_n;M)\xrightarrow{\delta}\cdots\to\check{C}^n(a_1,\dots,a_n;M)\to 0,\tag{9.1}$$

where

$$\check{C}^0(a_1,\dots,a_n;M)=M,\qquad \check{C}^i(a_1,\dots,a_n;M)=\prod_{H\subset\{1,\dots,n\},|H|=i}M_{a_H}$$

and

$$\delta(m)_{h_0,h_1,\dots,h_r}=\sum_i(-1)^i m_{h_0,h_1,\dots,\widehat{h_i},\dots,h_r}.$$

is the Čech differential.

Definition 9.2. We will denote by

$$H^i_{\underline{a}}(M)=H^i(a_1,\dots,a_n;M)$$

the i-th cohomology group of the complex $\check{C}^*(a_1,\dots,a_n;M)$: notice that $H^i(a_1,\dots,a_n;M)$ is an A-module.

Remark 9.3. Let $A=\bigoplus_i A^i$ be a fixed DG-ring and consider the scheme $Spec(Z^0(A))$. For every $Z^0(A)$-module F, we can consider the associated quasi-coherent sheaf $\widetilde{F}$ on $Spec(Z^0(A))$. Then, the complex $\check{C}^*(a_1,\dots,a_n;F)$ computes the local cohomology of $\widetilde{F}$ with support on the ideal generated by $a_1,\dots,a_n$.

Every permutation σ of $\{1,\dots,n\}$ induces a natural isomorphism (see e.g. [BS98])

$$\check{\sigma}\colon \check{C}^*(a_1,\dots,a_n;M)\to \check{C}^*(a_{\sigma(1)},\dots,a_{\sigma(n)};M).$$

For later use, we point out that, via the natural isomorphism $M_{a_1\cdots a_n}=M_{a_{\sigma(1)}\cdots a_{\sigma(n)}}$, the map

$$\check{\sigma}\colon \check{C}^n(a_1,\dots,a_n;M)\to \check{C}^n(a_{\sigma(1)},\dots,a_{\sigma(n)};M)$$

is simply given by multiplication by the signature of σ.

Moreover,

$$\check{C}^i(a_1,\dots,a_n;M)=\check{C}^i(a_1,\dots,a_n;A)\otimes_A M=\check{C}^i(a_1,\dots,a_n;Z^0(A))\otimes_{Z^0(A)} M\ .$$

Given any $b\in Z^0(A)$ and any integer i, we have a natural projection morphism of A-modules

$$\check{C}^i(a_1,\dots,a_n,b;M)\to \check{C}^i(a_1,\dots,a_n;M)\to 0\ .$$

The same argument used in [BS98, p. 86] shows that, if $b\in\sqrt{(a_1,\dots,a_n)}\subset Z^0(A)$, then the projection map $\check{C}^*(a_1,\dots,a_n,b;M)\to\check{C}^*(a_1,\dots,a_n;M)$ is a quasi-isomorphism in the category of complexes of A-modules.

Theorem 9.4. *The isomorphism class of the cohomology modules*

$$H^i(a_1,\dots,a_n;M)$$

depends only on the ideal $\sqrt{(a_1,\dots,a_n)}\subset Z^0(A)$. More precisely, if

$$\sqrt{(a_1,\dots,a_n)}=\sqrt{(b_1,\dots,b_m)}\subset Z^0(A),$$

then there exists a canonical *isomorphism*

$$H^i(a_1,\dots,a_n;M)\cong H^i(b_1,\dots,b_m;M).$$

Proof. We have

$$\begin{tikzcd}
\check{C}^*(a_1,\dots,a_n,b_1,\dots,b_m;M) \arrow[r, "\text{permutation}", "\simeq"'] \arrow[d] & \check{C}^*(b_1,\dots,b_m,a_1,\dots,a_n;M) \arrow[d] \\
\check{C}^*(a_1,\dots,a_n;M) & \check{C}^*(b_1,\dots,b_m;M)
\end{tikzcd}$$

where the vertical maps are the surjective quasi-isomorphisms induced by projections. □

The complex of A-modules $\check{C}^*(a_1,\dots,a_n;M)$ can be clearly interpreted as a double complex of $Z^0(A)$-modules. We will denote by $\mathbb{H}^*_{\underline{a}}(M)=\mathbb{H}^*(a_1,\dots,a_n;M)$ the cohomology of the associated total complex $\operatorname{tot}(\check{C}^*(a_1,\dots,a_n;M))$.

Thus, we have

$$\operatorname{tot}(\check{C}^*(a_1,\dots,a_n;M))=\operatorname{tot}(\check{C}^*(a_1,\dots,a_n;A))\otimes_A M=\operatorname{tot}(\check{C}^*(a_1,\dots,a_n;Z^0(A)))\otimes_{Z^0(A)}M$$

and, since $\check{C}^*(a_1,\dots,a_n;M)$ is a complex of finite length, we have two convergent spectral sequences: the former is

$$E_2^{i,j}=H^i(H^j(a_1,\dots,a_n;M))\Rightarrow \mathbb{H}^{i+j}(a_1,\dots,a_n;M),$$

and, since the localization is an exact functor, the latter is

$$E_2^{i,j}=H^i(a_1,\dots,a_n;H^j(M))\Rightarrow \mathbb{H}^{i+j}(a_1,\dots,a_n;M).$$

Lemma 9.5. *Assume that $Z^0(A)$ is a Noetherian ring; if $a_1,\dots,a_n$ is a regular sequence in $Z^0(A)$ and M is a complex of free $Z^0(A)$-modules, then*

$$\mathbb{H}^i(a_1,\dots,a_n;M)=H^{i-n}(H^n(a_1,\dots,a_n;M)),$$

for every $i\in\mathbb{Z}$.

Proof. As observed in Remark 9.3, for every $Z^0(A)$-module F, the complex $\check{C}^*(a_1,\ldots,a_n;F)$ computes the local cohomology with support on the ideal generated by $a_1,\ldots,a_n$ of the quasi-coherent sheaf $\widetilde{F}$ on $Spec(Z^0(A))$; therefore, $H^i(a_1,\ldots,a_n;Z^0(A))=0$ for every $i\neq n$ [Gr67, Thm. 3.8]. If F is free then also $H^i(a_1,\ldots,a_n;F)=0$ for every $i\neq n$, since it is a direct sum of copies of $H^i(a_1,\ldots,a_n;Z^0(A))$. □

Next, let us consider the functorial properties of local cohomology.

First, if $f\colon M\to N$ a morphism of A-modules, then f induces a morphism of complexes of A-modules

$$f_*\colon \check{C}^*(a_1,\ldots,a_n;M)\to \check{C}^*(a_1,\ldots,a_n;N)$$

and, therefore, morphisms

$$f_i\colon H^i(a_1,\ldots,a_n;M)\to H^i(a_1,\ldots,a_n;N),\qquad f_i\colon \mathbb{H}^i(a_1,\ldots,a_n;M)\to \mathbb{H}^i(a_1,\ldots,a_n;N).$$

Lemma 9.6. *In the notation above, if $f\colon M\to N$ is a quasi-isomorphism, then*

$$f_i\colon \mathbb{H}^i(a_1,\ldots,a_n;M)\to \mathbb{H}^i(a_1,\ldots,a_n;N)$$

is an isomorphism for every i.

Proof. Immediate consequence of the spectral sequence

$$H^j(a_1,\ldots,a_n;H^i(M))\Rightarrow \mathbb{H}^{i+j}(a_1,\ldots,a_n;M).$$

□

Let $g\colon A\to B$ be a morphism of DG-rings, then for every A-Module M, setting $N=M\otimes_A B$, we have natural maps $M\to N$, $M_{a_H}\to M\otimes_A B_{g(a_H)}$, and, therefore, a morphism of complexes

$$g_*\colon \check{C}^*(a_1,\ldots,a_n;M)\to \check{C}^*(g(a_1),\ldots,g(a_n);M\otimes_A B)$$

inducing morphisms of A-modules

$$g_*\colon H^i(a_1,\ldots,a_n;M)\to H^i(g(a_1),\ldots,g(a_n);M\otimes_A B).$$

Lemma 9.7. *If $g\colon A\to B$ is a quasi-isomorphism of DG-rings and M is a semifree A-module, then*

$$g_*\colon \mathbb{H}^i(a_1,\ldots,a_n;M)\to \mathbb{H}^i(g(a_1),\ldots,g(a_n);M\otimes_A B).$$

is an isomorphism for every i.

Proof. For every $a\in Z^0(A)$, we have a quasi-isomorphism $g\colon A_a\to B_{g(a)}$; then, since M is semifree, we also have a quasi-isomorphism

$$M_a=A_a\otimes_A M\to B_{g(a)}\otimes_A M.$$

By Lemma 9.6, this induce quasi-isomorphisms

$$\check{C}^i(a_1,\ldots,a_n;M)\to \check{C}^i(g(a_1),\ldots,g(a_n);M\otimes_A B),$$

$$\operatorname{tot}(C^*(a_1,\ldots,a_n;M))\to \operatorname{tot}(C^*(g(a_1),\ldots,g(a_n);M\otimes_A B)).$$

□

Remark 9.8. The calculus $\operatorname{Der}^*_{\mathbb{K}}(A,A)\times\Omega^*_A\to\Omega^*_A$ induced by the interior product and the de Rham differential commutes with local cohomology construction. In fact, since it commutes with localizations, we have for every $a_1,\ldots,a_n\in Z^0(A)$ a sequence of calculi

$$\operatorname{Der}^*_{\mathbb{K}}(A,A)\times C^j(a_1,\ldots,a_n,\Omega^*_A)\to C^j(a_1,\ldots,a_n,\Omega^*_A),\qquad j=0,\ldots,n,$$

commuting with Čech differentials and so inducing a sequence of calculi

$$\operatorname{Der}^*_{\mathbb{K}}(A,A)\times H^j(a_1,\ldots,a_n;\Omega^*_A)\to H^j(a_1,\ldots,a_n;\Omega^*_A),\qquad j=0,\ldots,n.$$

Remark 9.9. Since the construction of $H^i(a_1,\dots,a_n;M)$ commutes with localization, we have

$$H^i(a_1,\dots,a_n;M_b)=H^i(a_1,\dots,a_n;M)_b,\qquad \forall\, b\in Z^0(A),$$

and therefore, if $\mathcal{F}$ is any quasi-coherent sheaf on a DG-scheme $(T,\mathcal{A})$, and $Y\subset T$ is any closed subscheme of T, we have the local cohomology sheaves $\mathcal{H}^i_Y(T,\mathcal{F})$ defined locally as $H^i(a_1,\dots,a_n,\mathcal{F})$, where $a_1,\dots,a_n$ is a set of generators of the ideal sheaf of Y in T. This applies in particular for the quasi-coherent DG-sheaves $\Omega^k_{\mathcal{A}}$.

10. The derived cycle class of Z

Let us go back to the situation described in the introduction, i.e., $Z\subset X$ local complete intersection of codimension p, defined as the zero locus of a section f of a vector bundle $\pi\colon E\to U$ of rank p. As in [Bl72], we have the local cohomology sheaves $\mathcal{H}^i_Z(X,\Omega^j_X)$ and a canonical cycle class

$$\{Z\}\in\mathbb{H}^{2p}(X,\mathcal{H}^p_Z(X,\Omega^p_X)\xrightarrow{d}\mathcal{H}^p_Z(X,\Omega^{p+1}_X)\to\cdots),$$

where $\mathcal{H}^p_Z(X,\Omega^j_X)$ is a DG-sheaf concentrated in degree $p+j$. Recall that, since Z is a locally complete intersection and Ω^j_X is locally free, then $\mathcal{H}^i_Z(X,\Omega^j_X)=0$ for every $i\neq p$. The image of $\{Z\}$ under the injective map

$$\mathbb{H}^{2p}(X,\mathcal{H}^p_Z(X,\Omega^p_X)\xrightarrow{d}\mathcal{H}^p_Z(X,\Omega^{p+1}_X)\to\cdots)\hookrightarrow\mathbb{H}^{2p}(X,\mathcal{H}^p_Z(\Omega^p_X))=\Gamma(X,\mathcal{H}^p_Z(\Omega^p_X))$$

is the global section $\{Z\}'\in\Gamma(X,\mathcal{H}^p_Z(\Omega^p_X))=\Gamma(U,\mathcal{H}^p_Z(\Omega^p_U))$ given in local coordinates by

$$\{Z\}'=\frac{df_1\wedge\cdots\wedge df_p}{f_1\cdots f_p}\in\check{C}^p(f_1,\dots,f_p;\Omega^p_X),$$

where $f_1,\dots,f_p$ are the components of f with respect to a local trivialization of the bundle E. By local duality, the section $\{Z\}'$ is non trivial at every point of Z.

Similarly, working in the DG-scheme $(E,\mathcal{S})$, if $U^0\subset E$ is the image of the zero section $U\to E$, we have a canonical cycle class $\{U^0\}'\in\Gamma(E,\mathcal{H}^p_{U^0}(\Omega^p_{\mathcal{S}}))$, given in local coordinates by

$$\{U^0\}'=\frac{dz_1\wedge\cdots\wedge dz_p}{z_1\cdots z_p},$$

where $z_1,\dots,z_p$ are a set of linear coordinates in the fibers of the bundle E. The same proof of the classical case shows that $\{U^0\}'$ is independent from the choice of the coordinate system.

Since U^0 is smooth of codimension p in E and the DG-sheaves $\Omega^j_{\mathcal{S}}$ are semifree, we have $\mathcal{H}^i_{U^0}(E,\Omega^j_{\mathcal{S}})=0$ for every j and every $i\neq p$.

Definition 10.1. In the above notation, for every $j\geq 0$, we denote $\mathcal{K}^j=\pi_*\mathcal{H}^p_{U^0}(E,\Omega^j_{\mathcal{S}})$.

Every $\mathcal{K}^j$ is a quasi-coherent DG-sheaf on U.

Lemma 10.2. *For every $j\geq 0$, the section f induces a quasi-isomorphism of DG-sheaves on U*

$$\mathcal{K}^j\to\mathcal{H}^p_Z(U,\Omega^j_U)=\mathcal{H}^p_Z(X,\Omega^j_X),$$

with $\mathcal{H}^p_Z(U,\Omega^j_U)$ concentrated in degree $p+j$.

Proof. The question is local, therefore we may assume $U=Spec(P)$ affine and E trivial on U, i.e., $E=Spec(P[z_1,\dots,z_p])$. Thus

$$\Gamma(E,\mathcal{S})=S=P[z_1,\dots,z_p,y_1,\dots,y_p]\qquad\text{with}\quad \deg(y_i)=-1\quad d(y_i)=f_i-z_i.$$

The map, induced by f:

$$S\to P,\qquad y_i\mapsto 0,\quad z_i\mapsto f_i$$

is a quasi-isomorphism and then, by Lemma 9.7 we have isomorphisms of P-modules

$$\mathbb{H}^i_{U^0}(E,\Omega^j_S)\to \mathbb{H}^i_Z(U,\Omega^j_S\otimes_S P),$$

that, according to Lemma 9.5 gives a quasi-isomorphism of DG-sheaves

$$\mathcal{K}^j=\pi_*\mathcal{H}^p_{U^0}(E,\Omega^j_{\mathcal{S}})\to \mathcal{H}^p_Z(U,\Omega^j_{\mathcal{S}}\otimes_{\mathcal{S}}\mathcal{O}_U).$$

On the other hand, since U is smooth, by general properties of the cotangent complex, the natural map $\Omega^1_S\otimes_S P\to\Omega^1_P$ is a quasi-isomorphism, and also a homotopy equivalence since $\Omega^1_S\otimes_S P$ and Ω^1_P are semifree. This implies that every morphism $\Omega^j_S\otimes_S P\to\Omega^j_P$ is a homotopy equivalence; then, by Lemma 9.6 we get isomorphisms

$$\mathbb{H}^i_Z(U,\Omega^j_S\otimes_S P)\to \mathbb{H}^i_Z(U,\Omega^j_P)$$

and so a quasi-isomorphism of DG-sheaves

$$\mathcal{H}^p_Z(U,\Omega^j_{\mathcal{S}}\otimes_{\mathcal{S}}\mathcal{O}_U)\to \mathcal{H}^p_Z(U,\Omega^j_U).$$

□

It is immediate to observe that the map $\Gamma(U,\mathcal{K}^p)\to\Gamma(U,\mathcal{H}^p_Z(\Omega^p_U))$ sends the cycle class $\{U^0\}'$ into the cycle class $\{Z\}'$. In particular, $\{U^0\}'$ gives a non trivial cohomology class in $\mathbb{H}^{2p}(U,\mathcal{K}^p)$ and, since $\{U^0\}'$ is annihilated by the de Rham differential, it gives also a non trivial hypercohomology class

$$[\{U^0\}']\in\mathbb{H}^{2p}(U,\mathcal{K}^{\geq p})=\mathbb{H}^{2p}(U,\mathcal{K}^p\xrightarrow{d}\mathcal{K}^{p+1}\xrightarrow{d}\cdots).$$

11. Proof of the main theorem

Finally, we are ready for the proof of the main theorem. Recall that $\mathcal{K}^j=\pi_*\mathcal{H}^p_{U^0}(E,\Omega^j_{\mathcal{S}})$, $\mathcal{M}=\mathcal{D}er^*_{\mathcal{O}_U}(\operatorname{Sym}^*_{\mathcal{O}_U}(\mathcal{E}^\vee[1]\oplus\mathcal{E}^\vee),\operatorname{Sym}^*_{\mathcal{O}_U}(\mathcal{E}^\vee[1]\oplus\mathcal{E}^\vee))$ and $\mathcal{M}_\perp\subset\mathcal{M}$ is the subsheaf of derivations preserving the ideal generated by $\mathcal{E}^\vee$.

The de Rham differential induces a map $d\colon\mathcal{K}^*\to\mathcal{K}^*$ of degree $+1$ where $\mathcal{K}^*=\oplus_i\mathcal{K}^i=\oplus_i\pi_*\mathcal{H}^p_{U^0}(E,\Omega^i_{\mathcal{S}})$, inducing a structure of double differential graded sheaf on $\mathcal{K}^*$.

Then, according to Remark 9.8, the internal product induces a calculus

$$\mathcal{M}\times\mathcal{K}^*\to\mathcal{K}^*.$$

Passing to an affine open cover $\mathcal{U}$, by Lemma 7.9, we get a calculus

$$TW(\mathcal{U},\mathcal{M})\times TW(\mathcal{U},\mathcal{K}^*)\to TW(\mathcal{U},\mathcal{K}^*),\quad\text{where}\quad TW(\mathcal{U},\mathcal{K}^*)=\oplus_j TW(\mathcal{U},\mathcal{K}^j).$$

For simplicity of notation, we still denote by $\gamma\in TW(\mathcal{U},\mathcal{K}^p)$ the image of the canonical cycle class $\{U^0\}'$ under the natural map $\Gamma(U,\mathcal{K}^p)\to TW(\mathcal{U},\mathcal{K}^p)$ described in Lemma 3.4; we will also denote by $TW(\mathcal{U},\mathcal{K}^{\geq p})=\oplus_{j\geq p}TW(\mathcal{U},\mathcal{K}^j)$.

Next, consider the following four differential graded Lie algebras

$$L=\operatorname{Hom}^*_{\mathbb{K}}(TW(\mathcal{U},\mathcal{K}^*),TW(\mathcal{U},\mathcal{K}^*)),\qquad L(\gamma_\perp)=\{f\in L\mid f(\gamma)=0\},$$

$$\overline{L}=\{f\in L\mid f(TW(\mathcal{U},\mathcal{K}^{\geq p}))\subset TW(\mathcal{U},\mathcal{K}^{\geq p})\},\qquad \overline{L}(\gamma_\perp)=L(\gamma_\perp)\cap\overline{L},$$

they give a commutative diagram of inclusions

$$\begin{tikzcd}
\overline{L}(\gamma_\perp) \arrow[r, "\overline{\rho}"] \arrow[d, "j_\perp"] & \overline{L} \arrow[d, "j"] \\
L(\gamma_\perp) \arrow[r, "\rho"] & L.
\end{tikzcd}$$

Note that this diagram induces a morphism of DGLAs $TW(j_\perp)\to TW(j)$.

The morphism $\boldsymbol{l}$ induced by the Cartan homotopy
$$\boldsymbol{i}\colon TW(\mathcal{U},\mathcal{M}) \to L = \operatorname{Hom}^*_{\mathbb{K}}(TW(\mathcal{U},\mathcal{K}^*), TW(\mathcal{U},\mathcal{K}^*))$$
preserves the subcomplex $TW(\mathcal{U},\mathcal{K}^{\geq p}) = \oplus_{j\geq p} TW(\mathcal{U},\mathcal{K}^j)$, i.e., $\boldsymbol{l}(TW(\mathcal{U},\mathcal{M})) \subset \overline{L}$, and then Corollary 7.5 gives a canonical L_∞ morphism
$$\alpha_\infty\colon TW(\mathcal{U},\mathcal{M}) \dashrightarrow TW(j).$$
By Remark 2.2 the cohomology of $TW(j)$ is the same as the cohomology of
$$\operatorname{Hom}^*_{\mathbb{K}}(TW(\mathcal{U},\mathcal{K}^{\geq p}), TW(\mathcal{U},\mathcal{K}^{<p}))[-1], \quad \text{where} \quad TW(\mathcal{U},\mathcal{K}^{<p}) = \frac{TW(\mathcal{U},\mathcal{K}^*)}{TW(\mathcal{U},\mathcal{K}^{\geq p})}.$$
Analogously, since
$$\boldsymbol{i}(TW(\mathcal{U},\mathcal{M}_\perp)) \subset L(\gamma_\perp), \quad \text{and} \quad \boldsymbol{l}(TW(\mathcal{U},\mathcal{M}_\perp)) \subset \overline{L}(\gamma_\perp),$$
we have a canonical L_∞ morphism
$$\alpha_\infty\colon TW(\mathcal{U},\mathcal{M}_\perp) \dashrightarrow TW(j_\perp).$$
Therefore, we have a commutative diagram of L_∞ morphisms

(11.1)
$$\begin{tikzcd} TW(\mathcal{U},\mathcal{M}_\perp) \arrow[r, dashed, "\alpha_\infty"] \arrow[d, "\chi"] & TW(j_\perp) \arrow[d, "\rho"] \\ TW(\mathcal{U},\mathcal{M}) \arrow[r, dashed, "\alpha_\infty"] & TW(j) \end{tikzcd}$$

The cohomology of $TW(j_\perp)$ is the same as the cohomology of
$$\{f \in \operatorname{Hom}^*_{\mathbb{K}}(\mathcal{U}, TW(\mathcal{K}^{\geq p}), TW(\mathcal{U},\mathcal{K}^{<p}))[-1] \mid f(\gamma) = 0\}.$$
Since γ is non trivial in the cohomology of $TW(\mathcal{U},\mathcal{K}^{\geq p})$, we have that the inclusion $\rho\colon TW(j_\perp) \hookrightarrow TW(j)$ is injective in cohomology (see Example 2.3) and the cohomology of its homotopy fiber $TW(\rho)$ is the same as
$$\operatorname{Hom}^*_{\mathbb{K}}(\mathbb{K}\gamma, TW(\mathcal{K}^{<p}))[-2] = TW(\mathcal{U},\mathcal{K}^{<p})[2p-2],$$
where the last equality follows from the fact that γ has degree $2p$. Note that, by Lemma 2.1, the homotopy fiber of $\rho\colon TW(j_\perp) \hookrightarrow TW(j)$ is homotopy abelian. We are now ready to prove the main result of this paper.

Theorem 11.1. *In the above setup, the composition of the semiregularity map*
$$H^1(Z, N_{Z|X}) \xrightarrow{\pi} H^{p+1}(X, \Omega^{p-1}_X)$$
with the natural map
$$H^{p+1}(X, \Omega^{p-1}_X) \to \mathbb{H}^{2p}(X, \Omega^0_X \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{p-1}_X)$$
annihilates every obstruction to embedded deformations of Z in X. If the Hodge-de Rham spectral sequence of X degenerates at level E_1 (e.g. if X is smooth proper [DI87]*), then π also annihilates every obstruction.*

Proof. The Example 6.7 applied to the commutative square (11.1) gives an L_∞ morphism of homotopy fibers
$$\alpha_\infty\colon TW(\chi) \dashrightarrow TW(\rho)$$
and therefore a natural transformation of the associated deformation functors
$$\alpha_\infty\colon \operatorname{Def}_{TW(\chi)} \to \operatorname{Def}_{TW(\rho)}.$$
By Theorem 5.4 the functor $\operatorname{Def}_{TW(\chi)}$ is isomorphic to the functor $\operatorname{Hilb}_{Z|X}$ of embedded deformations; then, the obstruction map associated with α_∞ is
$$\alpha_\infty\colon H^2(TW(\chi)) = H^1(Z, N_{Z|X}) \to H^2(TW(\rho)) = H^{2p}(TW(\mathcal{U},\mathcal{K}^{<p}))$$

and it sends obstructions to embedded deformations to obstructions of the functor $\operatorname{Def}_{TW(\rho)}$. Since $TW(\rho)$ is homotopy abelian, the functor $\operatorname{Def}_{TW(\rho)}$ is unobstructed and this proves that the above map annihilates every obstruction.

Next, in view of the surjective quasi-isomorphisms of quasi-coherent sheaves $\mathcal{K}^j \to \mathcal{H}^p_Z(X,\Omega^j_X)$, given by Lemma 10.2, the cohomology of $TW(\mathcal{U},\mathcal{K}^{<p})$ is isomorphic to the hypercohomology of the complex

$$TW(\mathcal{U},\mathcal{H}^p_Z(\Omega^0_X)\xrightarrow{d}\cdots\xrightarrow{d}\mathcal{H}^p_Z(\Omega^{p-1}_X))$$

and therefore it is also isomorphic to the hypercohomology over U of the complex of sheaves

$$\mathcal{H}^p_Z(\Omega^0_X)\xrightarrow{d}\cdots\xrightarrow{d}\mathcal{H}^p_Z(\Omega^{p-1}_X).$$

Since Z is a locally complete intersection, the spectral sequence of local cohomology degenerates. Thus,

$$\mathbb{H}^*(U,\mathcal{H}^p_Z(U,\Omega^0_X)\xrightarrow{d}\cdots\xrightarrow{d}\mathcal{H}^p_Z(U,\Omega^{p-1}_X))=\mathbb{H}^*_Z(U,\Omega^0_X\xrightarrow{d}\cdots\xrightarrow{d}\Omega^{p-1}_X)$$

and, therefore, the map

$$\alpha_\infty\colon H^1(Z,N_{Z|X})\to H^2(TW(\rho))=\mathbb{H}^{2p}_Z(U,\Omega^0_X\xrightarrow{d}\cdots\xrightarrow{d}\Omega^{p-1}_X)$$

annihilates the obstructions to embedded deformations. By general results about local cohomology of finite complexes of sheaves, we have a natural map

$$\mathbb{H}^{2p}_Z(U,\Omega^0_X\xrightarrow{d}\cdots\xrightarrow{d}\Omega^{p-1}_X)=\mathbb{H}^{2p}_Z(X,\Omega^0_X\xrightarrow{d}\cdots\xrightarrow{d}\Omega^{p-1}_X)\to\mathbb{H}^{2p}(X,\Omega^0_X\xrightarrow{d}\cdots\xrightarrow{d}\Omega^{p-1}_X).$$

In order to conclude the proof, it is now sufficient to prove the commutativity of the diagram

$$\begin{tikzcd}
H^1(Z,N_{Z|X}) \arrow[r, "\alpha_\infty"] \arrow[d, "\pi"] & \mathbb{H}^{2p}_Z(U,\Omega^0_X\xrightarrow{d}\cdots\xrightarrow{d}\Omega^{p-1}_X) \arrow[d] \\
H^{p+1}(X,\Omega^{p-1}_X) \arrow[r] & \mathbb{H}^{2p}(X,\Omega^0_X\xrightarrow{d}\cdots\xrightarrow{d}\Omega^{p-1}_X).
\end{tikzcd}$$

According to the definition of the semiregularity map given in the introduction, it is sufficient to prove that we have a commutative diagram

$$\begin{tikzcd}
H^1(Z,N_{Z|X}) \arrow[r, "\lrcorner\{Z\}'"] \arrow[dr, "\alpha_\infty"'] & H^{p+1}_Z(U,\Omega^{p-1}_X) \arrow[d] \\
 & \mathbb{H}^{2p}_Z(U,\Omega^0_X\xrightarrow{d}\cdots\xrightarrow{d}\Omega^{p-1}_X).
\end{tikzcd}$$

Next, the composition of the linear part of the L_∞ morphism $\alpha_\infty\colon TW(\chi)\dashrightarrow TW(\rho)$ with the quasi-isomorphism $TW(\rho)\to TW(\mathcal{U},\mathcal{K}^{<p})[2p-2]$ described in Remark 2.2 is induced by the internal product with the canonical cycle class $\{U^0\}'$ and the conclusion follows from the fact that $\{U^0\}'$ maps onto $\{Z\}'$ via the morphism $\mathcal{K}^p\to\mathcal{H}^p_Z(U,\Omega^p_X)$. $\square$

Remark 11.2. An essentially equivalent proof, but technically easier, shows that the semiregularity map annihilates obstructions to embedded locally trivial deformations of a local complete intersection subvariety Z, without the set-up assumption of the introduction. To this end it is sufficient to take $(E,\mathcal{S})=(X,\mathcal{O}_X)$, $\mathcal{M}=\Theta_X$ the tangent sheaf of X and $\mathcal{M}_\perp=\Theta_X(-\log(Z))$ the subsheaf of derivations of $\mathcal{O}_X$ preserving the ideal of Z, see [Ia10].

References

[Ar76] M. Artin: *Deformations of singularities.* Tata Institute of Fundamental Research, Bombay (1976). 3, 11

[AF91] L.L. Avramov and H.-B. Foxby: *Homological dimensions of unbounded complexes.* J. Pure Appl. Algebra **71** (1991) 129-155. 5

[Bl72] S. Bloch: *Semi-regularity and de Rham cohomology.* Invent. Math., **17** (1972), 51-66. 1, 3, 25

[BS98] M.P. Broadmann and R.Y. Sharp: *Local cohomology.* Cambridge University Press (1998). 22, 23

[BF03] R-O. Buchweitz and H. Flenner: *A semiregularity map for modules and applications to deformations.* Compos. Math., **137** (No. 2), (2003), 135-210. 3

[CK01] I. Ciocan-Fontanine and M.M. Kapranov: *Derived Quot Schemes.* Ann. Sci. École Norm. Sup. (4) **34** (2001) 403-440. `arXiv:math/9905174v2 [math.AG]` 2, 13

[CG08] X.Z. Cheng and E. Getzler: *Homotopy commutative algebraic structures.* J. Pure Appl. Algebra **212** (2008) 2535-2542; `arXiv:math.AT/0610912`. 7

[DI87] P. Deligne and L. Illusie: *Relévements modulo p^2 et décomposition du complexe de de Rham.* Invent. Math. **89** (1987) 247-270. 2, 27

[EZ50] S. Eilenberg and J.A. Zilber: *Semi-simplicial complexes and singular homology.* Ann. of Math. (2) **51** (1950) 499-513. 7

[FaMa98] B. Fantechi and M. Manetti: *Obstruction calculus for functors of Artin rings, I.* J. Algebra, **202**, (1998), 541-576. 17

[FHT01] Y. Félix, S. Halperin and J. Thomas: *Rational homotopy theory.* Graduate texts in mathematics **205**, Springer-Verlag, New York Berlin, (2001). 7

[FiMa07] D. Fiorenza and M. Manetti: *L_∞ structures on mapping cones.* Algebra Number Theory, **1**, (2007), 301-330; `arXiv:math.QA/0601312`. 7, 8, 14, 17

[FiMa09] D. Fiorenza and M. Manetti: *A period map for generalized deformations.* J. Noncommut. Geom. **3** (2009) 579-597; `arXiv:0808.0140v1 [math.AG]`. 17

[FiMa] D. Fiorenza and M. Manetti: *In preparation.* 3

[FIM09] D. Fiorenza, D. Iacono and E. Martinengo: *Differential graded Lie algebras controlling infinitesimal deformations of coherent sheaves.* J. Eur. Math. Soc. (JEMS) **14** (2012), no. 2, 521-540; `arXiv:0904.1301v3 [math.QA]`. 3, 11

[FMM10] D. Fiorenza, M. Manetti and E. Martinengo: *Semicosimplicial DGLAs in deformation theory.* Communications in Algebra **40**, Iss. 6, (2012) 2243-2260; `arXiv:0803.0399v2 [math.QA]`. 10

[Fu03] K. Fukaya: *Deformation theory, homological algebra and mirror symmetry.* Geometry and physics of branes (Como, 2001), Ser. High Energy Phys. Cosmol. Gravit., IOP Bristol, (2003), 121-209. 14

[Get04] E. Getzler: *Lie theory for nilpotent L_∞-algebras.* Ann. of Math. **170** (1), (2009) 271-301; `arXiv:math/0404003v4`. 7, 14

[GM88] W.M. Goldman and J.J. Millson: *The deformation theory of representations of fundamental groups of compact Kähler manifolds.* Publ. Math. I.H.E.S. **67** (1988) 43-96. 10

[GH78] Ph. Griffiths and J. Harris: *Principles of algebraic geometry.* Wiley Classics Library, Wiley, New York, (1978). 2

[Gr67] A. Grothendieck: *Local cohomology.* Lecture notes in Mathematics **41** (1967). 4, 24

[Hin97] V. Hinich: *Descent of Deligne groupoids.* Internat. Math. Res. Notices (1997), no. 5, 223-239. 3, 10, 11

[Ia07] D. Iacono:*Algebre di Lie Differenziali Graduate e Deformazioni di Mappe Olomorfe*. Bollettino Unione Matematica Italiana A, **10** (2) (2007) 251-254. 1

[Ia10] D. Iacono: *Deformations of algebraic subvarieties.* Rend. Mat. Appl. (7) **30** (2010), no. 1, 89-109; `arXiv:1003.3333`. 11, 28

[Ia11] D. Iacono: *A semiregularity map annihilating obstructions to deforming holomorphic maps.* Canad. Math. Bull. **54** (2011), no. 3, 472-486; `arXiv:0707.2454`. 1

[IM10] D. Iacono and M. Manetti: *An algebraic proof of Bogomolov-Tian-Todorov theorem* Deformation Spaces. Vol. 39 (2010) 113-133; `arXiv:0902.0732v2 [math.AG]`. 2, 6, 8, 20

[Ka01] M. Kapranov: *Injective resolutions of BG and derived moduli spaces of local systems.* J. Pure Appl. Algebra **155** (2001) 167-179. 2

[Ke05] B. Keller: *Deformation quantization after Kontsevich and Tamarkin.* Déformation, quantification, théorie de Lie, Panor. Synthèses **20**, Soc. Math. France, Paris (2005) 1962. 18

[Kon94] M. Kontsevich: *Topics in algebra-deformation theory.* Berkeley notes (1994). 9, 10

[Kon03] M. Kontsevich: *Deformation quantization of Poisson manifolds, I.* Letters in Mathematical Physics **66** (2003) 157-216; `arXiv:q-alg/9709040`. 14

[KT11] M. Kool and R. P. Thomas: *Reduced classes and curve counting on surfaces I: theory, II: calculations.* arXiv:1112.3069v1 [math.AG], arXiv:1112.3070v1 [math.AG]. 2

[LM95] T. Lada and M. Markl: *Strongly homotopy Lie algebras.* Comm. Algebra **23** (1995) 2147-2161; arXiv:hep-th/9406095. 14, 18

[LS93] T. Lada and J. Stasheff: *Introduction to sh Lie algebras for physicists.* Int. J. Theor. Phys. **32** (1993) 1087-1104; arXiv:hep-th/9209099. 14

[MPT10] D. Maulik, R. Pandharipande and R.P. Thomas: *Curves on K3 surfaces and modular forms,* with an Appendix by A. Pixton. J. Topol., **3** (2010) 937-996; arXiv:1001.2719. 2

[Ma02] M. Manetti: *Extended deformation functors.* Int. Math. Res. Not. **14** (2002) 719-756; arXiv:math.AG/9910071. 2, 14, 17

[Ma04a] M. Manetti: *Cohomological constraint to deformations of compact Kähler manifolds.* Adv. Math. **186** (2004) 125-142; arXiv:math.AG/0105175. 14

[Ma04b] M. Manetti: *Lectures on deformations of complex manifolds.* Rend. Mat. Appl. (7) **24** (2004) 1-183; arXiv:math.AG/0507286. 4, 14

[Ma07] M. Manetti: *Lie description of higher obstructions to deforming submanifolds.* Ann. Sc. Norm. Super. Pisa Cl. Sci., **6**, (2007) 631-659; arXiv:math.AG/0507287v1. 1, 8, 9, 11

[Ma09] M. Manetti: *Differential graded Lie algebras and formal deformation theory.* In *Algebraic Geometry: Seattle 2005.* Proc. Sympos. Pure Math. **80** (2009) 785-810. 10, 17

[NaA87] V. Navarro Aznar: *Sur la théorie de Hodge-Deligne.* Invent. Math. **90** (1987) 11-76. 7

[OSS80] C. Okonek, M. Schneider and H. Spindler: *Vector bundles on complex projective spaces.* Birkhäuser Verlag (1980). 2

[Pr12] J.P. Pridham: *Semiregularity as a consequence of Goodwillie's theorem.* arXiv:1208.3111v1 [math.AG]. 3

[Qui69] D. Quillen: *Rational homotopy theory.* Ann. of Math. **90** (1969) 205-295. 15

[SS79] M. Schlessinger and J. Stasheff: *Deformation Theory and Rational Homotopy Type.* Preprint (1979). 14

[Se06] E. Sernesi: *Deformations of Algebraic Schemes.* Grundlehren der mathematischen Wissenschaften, **334**, Springer-Verlag, New York Berlin, (2006). 3, 11, 17

[TT05] D. Tamarkin and B. Tsygan: *The ring of differential operators on forms in noncommutative calculus.* Graphs and patterns in mathematics and theoretical physics, Proc. Sympos. Pure Math., vol. 73, Amer. Math. Soc., Providence, RI, 2005, 105-131. 19

[We94] C.A. Weibel: *An introduction to homological algebra.* Cambridge Studies in Advanced Mathematics **38**, Cambridge Univesity Press, Cambridge, (1994). 7

[Whi57] H. Whitney: *Geometric integration theory.* Princeton University Press, Princeton, N. J., (1957). 7

[Ye11] A. Yekutieli: *MC Elements in Pronilpotent DG Lie Algebras.* arXiv:1103.1035v3 [math.QA] 9, 10

Università degli Studi di Bari,
Dipartimento di Matematica,
Via E. Orabona 4, I-70125 Bari, Italy.
E-mail address: iacono@dm.uniba.it
URL: www.dm.uniba.it/~iacono/

Università degli studi di Roma "La Sapienza",
Dipartimento di Matematica "Guido Castelnuovo",
P.le Aldo Moro 5, I-00185 Roma, Italy.
E-mail address: manetti@mat.uniroma1.it
URL: www.mat.uniroma1.it/people/manetti/